Eq. 4 and compute values for the percentage of reinforcing steel at cracking moments which are closer to the conditions of their actual practice. See Table 3.

2. REVIEW OF DEFLECTION CALCULATION PROCEDURES

Table 1 tabulates the steps in the procedure for computing deflection for beams of homogeneous materials, the existing procedure for reinforced concrete beams and the proposed procedure for reinforced concrete beams. A study of the procedures indicates three areas in which deflection computations for reinforced concrete are more complicated than for beams of other materials.

(1) Beams of structural steel or timber are usually (but not always) statically determinate whereas concrete beams are usually continuous, thus statically indeterminate. Continuity (regardless of the material from which the beam is constructed) leads to more computational effort to determine the correct distribution of moments. Procedures for simplifying this computation considerably will be discussed in the next section.

(2) Determination of the flexural stiffness of a concrete member involves long and tedious calculations working from basic principles and even longer calculations to determine the proper basis on which to make the stiffness calculations. By contrast, the flexural stiffness for beams of other materials is frequently tabulated. If tabulated values are unavailable, the flexural stiffness computations are no more difficult than for concrete beams and frequently much simpler. Procedures for simplifying this computation for concrete beams are discussed later.

(3) The effects of time dependent strains must be accounted for. The ACI Building Code recommends a simple multiplier of the short term deflections to account for long term deflections and it is difficult to visualize further simplification from this procedure. Therefore this step in the simplified procedure will not be discussed further.

3. SIMPLIFIED COMPUTATIONS

3.1 Estimation of Continuity Effects

The classical equation for deflection of a prismatic beam is:

$$a = \beta_a W l^3/EI \qquad \text{Eq. (1)}$$

where β_a = 1/384 for a fixed end beam and 5/384 for a simple beam with uniform load. Note that the factor β_a depends upon a precise knowledge of the loading as well as end fixity and varies by a factor of 5 for these two conditions. If the end fixity condition and loading pattern are different from tabulated conditions in standard references, deflection must be computed by more lengthy procedures.

Eq. (1) can be recast as follows:

$$a = \beta_b M l^2/EI \qquad \text{Eq. (2)}$$

where M is the moment at midspan. While β_b also depends upon loading conditions and end fixity, it only varies from 3/48 for a fixed end beam to 5/48 for a simple beam with uniform load. It may be noted that β_b varies by a factor of 1.67 between these two conditions. Due to the very narrow range in which β_b varies, it is possible to estimate an appropriate factor by inspection without computing the degree of fixity. Therefore, the deflection equation is recast in the following form:

$$a = \beta (5/48) M l^2/EI \qquad \text{Eq. (3)}$$

where M is the midspan moment at service loads, taken from the design for strength. (Note that this must be the actual moment and not an inflated moment which might arise from the use of standard moment factors such as those in Section 8.4 of the ACI Building Code (2).) And where β is a factor taken from Table 2 for typical conditions. Other conditions can be interpolated between the tabulated conditions, (but not extrapolated) using the following relationships:

for uniformly distributed loads, $\beta = 0.4 + 0.6\ (M/M_o)$ — Eq. (4)

for a concentrated load at midspan, $\beta = 0.8\ (M/M_o)$ — Eq. (5)

where M_o is the total static moment at service loads.

3.2 Estimation of Flexural Stiffness

Using the proposed simplified procedure, the required percentage of midspan tensile reinforcing steel is compared to a limiting value in Table 3. The required steel percentage must be computed on the basis of the steel required for the actual moment and not the actual steel furnished which in many cases is substantially larger. Table 3 is based on the previously stated assumption that the gross uncracked flexural rigidity can be used up to the cracking moment. The effective flexural rigidity is close to the flexural rigidity of the cracked section (i.e. 1/8 of the difference between the flexural rigidity of the cracked and uncracked sections, or less), when the actual moment is twice the cracking moment or more. For such high moments, the effective flexural rigidity can be approximated by the flexural rigidity of the cracked section. For intermediate values of the actual moment, the effective flexural rigidity can be approximated by 60% of the uncracked flexural rigidity.

The values in Table 3 were determined in the following manner. The percentage of principal reinforcement indicating the line of demarcation between an uncracked and a partially cracked cross-section can be computed by equating the moment at first cracking to the moment capacity of the reinforcing steel. Strength design as required by the 1971 ACI Building Code (2) is assumed. Thus:

$$f_r b_w h^2 \, \gamma_1/6 = M_{cr} = \rho_{cr} b_w \left[(d/h)h\right]^2 \left[0.9 \, f_y/\gamma_2\right](1 - a/2d)$$

and,
$$\rho_{cr} = \frac{\gamma_1 \; \gamma_2}{5.4\,(1 - a/2d)\,(d/h)^2} \times \frac{fr}{fy} \qquad \text{Eq. (6)}$$

in which,

ρ_{cr} = ratio of nonprestressed tension reinforcement required to resist a design moment equal to the cracking moment, times the average load factor.

f_r = modulus of rupture of concrete.

γ_1 = ratio of section modulus for an uncracked tee beam to section modulus for the stem alone of the uncracked tee beam.

γ_2 = average load factor.

$(1 - a/2d)$ = ratio of internal moment arm to the effective depth, d.

and other symbols are as defined in the ACI Building Code (2).

In determining the values to be used in Table 3, each of the factors in Eq. (6) needs to be analyzed for its effect on the percentage of reinforcing steel required. The evaluation for values used in Table 3 follows:

The value of γ_1 as a function of the flange area of the tee beam is plotted in Fig. 1. When the flange area is equal to the web area, γ_1 is approximately equal to 1.5 for the bottom fiber and 3.25 for the top fiber.

A straight line variation for γ_1 can be assumed for other ratios of flange area to web area, as a first approximation. For greater refinement, refer to Fig. 1 or compute the actual value.

The average load factor, γ_2, has been assumed equal to 1.55 [(1.4 + 1.7)/2] in the preparation of Table 3. This is equivalent to a condition of live load equal to dead load. While the actual factor can vary almost from 1.4 (for all dead load) to 1.7 (for all live load), the actual factor will almost always be within 5% of 1.55.

The ratio of internal lever arm, (1 - a/2d), has been calculated for each type of member at the cracking moment. This results in the following values used: 0.95 for rectangular sections, 0.99 for tee beams with tension in the stem, and 0.82 for tee beams with tension in the flange.

The value of d/h as a function of concrete cover and overall thickness (h) of the member is shown in Fig. 2. A value of 0.83 has been selected as being most representative. Almost all practical conditions will fall within 10% of this value and most will fall within 5%.

If a conservatively high value of computed deflection is desired, a low value of modulus of rupture (as suggested in the ACI Building Code) should be used. However, a value of 550 psi (38.7 kg/cm^2) has been selected for the modulus of rupture of nominal 3000 psi concrete in preparing Table 3. In the author's experience, using this value results in more accurate estimates of the actual deflection for reinforced concrete beams than using a lower value for the modulus of rupture.* It is equivalent to $9.3\sqrt{1.15 f'_c}$ for 3000 psi concrete ($2.5\sqrt{1.15 f'_c}$ for 211 kg/cm^2 concrete). A still higher value of the modulus should be used if a conservatively low value of computed deflection is desired.

In selecting the proper value of the modulus of rupture for deflection computations, a number of factors must be considered. First, the flexural stiffness depends more on the <u>average</u> modulus of rupture than on the <u>minimum</u> modulus of rupture since a single random crack occurring at very low loads will not affect the deflection of a beam very much. Second, in most beams, the average concrete strength will be higher than the minimum specified. Third, most beams will not experience their full design live load until after the concrete has gained more than the 28 day strength. Of course, the structural engineer should be aware of conditions in which these factors are not true.

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*Use of a high value for the modulus of rupture may be somewhat controversial. It is beyond the scope of this paper to discuss the merits of this proposal. However, engineers may use any desired value of f_r by substitution in Eq. (6).

The yield point of the reinforcing steel, f_y, is taken equal to 60,000 psi (4220 kg/cm^2).

To visualize the significance of Table 3, Figs. 3, 4 and 5 have been plotted for rectangular sections, tee beams with tension in stem (positive moment) and tee beams with tension in flange (negative moment), respectively. These figures plot the ratio of effective flexural rigidity to the gross uncracked flexural rigidity against the percentage of principal tensile steel required. The abscissa can also be stated in terms of the cracking moment. To plot these curves, it is necessary to assume a value for the ratio of the moment of inertia for an uncracked tee beam to the moment of inertia for the stem of the uncracked tee beam alone (γ_3). A value of γ_3 equal to 2.0 was selected as the most representative when the area of the flange of the tee beam equals the area of the stem. See Fig. 6.

4. ERROR

The method for computing deflections recommended by ACI Committee 435 (1) and adopted by the ACI Building Code (2), on which the simplifying procedures are based, in itself has some error compared to experimental data (5). As compared to this method, the proposed simplified procedures, of necessity, introduce some additional error. This additional error can be evaluated as follows:

In allowing for the effects of continuity, a very bad estimate could result in error of 20% to 35% in the computed deflection. (For example, if one estimates $\beta = 1.0$ for a beam with one end continuous when the correct value is $\beta = 0.74$, the error is 35%). However, in most cases, an experienced engineer should be able to estimate the value of β within 10% of the correct value for elastic, prismatic beams. If β is computed using Eq. (4) or (5), the maximum error introduced will be less than 13% and will average less than 8%. See Fig. 7.

The most serious error that will be introduced by this procedure is failure to use an accurate value for the midspan moment. However, the same error would be introduced in more lengthy procedures if inaccurate moments are used. It is not uncommon to use a midspan moment 50% higher than the actual moment or more when using standard moment coefficients or a limited moment distribution procedure. The engineer must resist the temptation to use these approximate but higher moments.

In computing the flexural stiffness, no error compared to the ACI Code (2) procedure will be introduced when the cross-section is uncracked. When the moment is larger than the cracking moment, the simplified procedure yields a variable error depending upon the magnitude of the moment. See Fig. 8. For those who desire greater accuracy, a variable coefficient of the gross flexural rigidity can be selected from Figs. 3, 4 or 5 instead of the coefficient of 0.6 recommended in Table 3. If this is done, the largest error compared to the ACI Code (2) procedure should be less than 10%.

The error, compared to the ACI Code (2) procedure, introduced by an inaccurate assessment of the percentage of principal tensile reinforcement required by the cracking moment can be studied by reference to Fig. 8. If the actual cracking moment is lower than the assumed value, the error for a moment slightly less than the assumed M_{cr} will be increased and the error for a moment slightly more than the assumed M_{cr} will be decreased. If the cracking moment is higher than assumed, a similar situation will prevail. A similar situation also prevails at twice the assumed value of M_{cr}. The net effect is to change the range in which the maximum error occurs but without changing the magnitude of the maximum error itself significantly.

Furthermore, other factors beyond the control of the structural engineer will affect the magnitude of the cracking moment more than the simplifications suggested here. For example, overloading the structure at an early age may crack it thoroughly even though it would be uncracked otherwise.

Another pertinent question is the maximum error that can be expected from the introduction of error from three separate sources. That is, 1) error from the method recommended by the ACI Building Code compared to experimental data, 2) error resulting from the recommended simplification to account for continuity, compared to the ACI Code procedure, and 3) error resulting from simplified computation of the flexural rigidity, compared to the ACI Code procedure.

If it is conservatively assumed that, for each source, at least 50% of the value of the maximum error will be reached in 50% of the cases and that the errors will be normally distributed above and below the actual value of deflection in the structure, then the odds that the total error will be at least 50% of the sum of the maximum error from each of the three sources will be one chance in 64 or less than 2%. The odds that the total error from all three sources would reach 100% of their sum would be considerably smaller. Thus, it appears that engineers need not fear errors introduced by the approximate procedure provided that the procedure is used consistently so as to distribute potential errors uniformly in each direction and provided that a total maximum error in deflection computations of about 50% is acceptable. In most instances such error is more than acceptable since computed deflections will be small and well within tolerable limits. In instances when the approximate procedure reveals deflections that may be critical, more precise procedures can be used with little loss of effort for having used the approximate procedure.

5. EXAMPLE

The uniformly loaded beam in Figs. 9 and 10 has a midspan moment of 3330 k-in (38,400 m-kg) which represents 70% of the total static moment. The required steel percentage is 1.7%. Estimate the short term deflection. (This is example 2-2 from Reference 6).

Estimate β = 0.8 from Table 2. (Or compute $\beta = 0.4 + 0.6 \times .7 = 0.82$ from Eq. (4)). Since the required steel percentage is greater than 1.2% from Table 3, use the cracked flexural rigidity. Using procedures outlined in Reference 6, the cracked flexural rigidity, $EI_{cr} = 10^5 \beta_{cr} bd^3$ where $\beta_{cr} = 0.78$ from Fig. 2-2 of Ref. 6 for b = 85.5 inches (217 cm) and $E_c = 3.5 \times 10^6$ psi (2.45×10^5 kg/cm^2). Thus $EI_{cr} = 10^5 \times 0.78 \times 85.5\ (21.25)^3 = 64 \times 10^9$ lb-in^2 (193×10^9 kg-cm^2). Alternatively, $EI_{cr} = E_s A_s (1-k)jd^2$ from Reference 1. Using classical equations, graphs or tabular data, k = 0.2157, j = .928 and $EI_{cr} = 29 \times 10^6 \times 6.68\ (1 - .2157)\ .928 \times (21.25)^2 = 64 \times 10^9$.

The short term deflection, from Eq. (3),

$$a_i = 0.8\ (5/48)\ 3{,}330{,}000\ (27 \times 12)^2 / 64 \times 10^9 \text{ lb-in}^2$$

$$= 0.45 \text{ inches } (1.1 \text{ cm})$$

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NOTATION

a = computed deflection.

A_f = area of flange of tee beam, see Fig. 1

A_w = area of web of tee beam, see Fig. 1

h_f = overall thickness of flange of tee beam.

M = midspan moment at service loads, taken from the design for strength.

M_o = total static moment at service loads (e.g. $M_o = wl^2/8$ for uniform load).

β = deflection factor, see Eq. (3)

β_a = deflection factor, see Eq. (1)

β_b = deflection factor, see Eq. (2)

γ_1 = ratio of section modulus for an uncracked tee beam to section modulus for the stem of the uncracked tee beam alone.

γ_2 = average load factor.

γ_3 = ratio of moment of inertia for an uncracked tee beam to moment of inertia for the stem of the uncracked tee beam alone.

ρ_{cr} = ratio of nonprestressed tension reinforcement required to resist a design moment equal to the cracking moment.

Other symbols used in this paper are identified in the ACI Building Code (2).

TABLE 1
Step by Step Procedures for Computing Deflection

Procedure for Beams of Homogeneous Materials	EXISTING Procedure for Reinforced Concrete Beams	PROPOSED Procedure for Reinforced Concrete Beams
1. Compute loading	1. Compute loading	1. Compute loading
2. Compute midspan & end moments	2. Compute midspan & end moments	2. Compute midspan moments

The first two steps must be performed in the design for strength. Values obtained may be used in the deflection analysis.

3. Compute moment of inertia (I) at midspan.	3. Compute moment of inertia (I) at midspan based on gross concrete cross section.	3. Compute ρ_w at midspan and select procedure for computing flexural rigidity from Table 2.
	4. Compute section modulus of gross concrete cross section at midspan.	4. Compute effective moment of inertia (I) at midspan.
	5. Compute flexural stresses at mid-span and compare them to modulus of rupture of the concrete.	
	6. If tensile flexural stress is greater than modulus of rupture, compute moment of inertia (I) at midspan based on cracked concrete cross section.	
	7. Determine the effective moment of inertia by interpolating between cracked and gross moment of inertia.	