$$\int_0^{2\pi} qR\,d\phi + w\left[\frac{1}{2} r\,d\phi (R - r) + \frac{1}{2}(R - r)^2 d\phi \frac{2}{3}\right] = \int_0^{2\pi} \frac{R\,d\phi}{R - r}(1 + k_m)m \qquad (3)$$

Substituting the value of q from Eq. 2 into Eq. 3, letting $\rho = \frac{R}{r}$ and $A_{col.} = \pi r^2$, one obtains

$$\frac{P}{m(1 + k_m)} = \frac{wA_{slab}}{m(1 + k_m)} = \frac{6\pi\rho \frac{A_{slab}}{A_{col.}}}{3(\rho - 1)\frac{A_{slab}}{A_{col.}} - (\rho^3 - 1)} \qquad (4)$$

The value of ρ in the above equation can be found by taking advantage of the fact that yield line theory is an upper bound theory. Therefore the smallest possible value of $\frac{P}{m(1 + k_m)}$ must be the correct one. Differentiating Eq. 4 with respect to ρ and setting the right hand side equal to zero, one obtains

$$\rho = \sqrt[3]{\frac{3}{2}\frac{A_{slab}}{A_{col.}} - \frac{1}{2}} \qquad (5)$$

Eq. 4 and 5 are presented in graphical form in Fig. 8-4, with Johansen's interior column formulas shown as dashed lines for comparison.

Case II–Interior, Square Column

A square column with length of side s, supports the load acting on A_{slab}. The other parameters are as for Case I. A geometrically admissible yield line pattern is shown in Fig. 8-5. The distance b' is to be determined. The shear q on the perimeter of the yield line pattern will be

$$q = \frac{P - w(s + 2b')^2}{4(s + 2b')} \qquad (6)$$

Dropping the rim of the square pattern a unit distance, one can again write the virtual work equation (for one quarter of the pattern):

$$q(s + 2b') + 2w\frac{b'^2}{2}\frac{2}{3} + \frac{1}{2} w\,s\,b' = m(1 + k_m)\frac{1}{b'}(s + 2b') \qquad (7)$$

Substituting q from Eq. 6 into Eq. 7, letting $\beta = \frac{b'}{s}$ and letting $A'_{col.} = s^2$, one obtains

$$\frac{P}{m(1 + k_m)} = \frac{wA_{slab}}{m(1 + k_m)} = \frac{4(2\beta + 1)\frac{A_{slab}}{A'_{col.}}}{\beta\left(\frac{A_{slab}}{A'_{col.}} - \frac{4}{3}\beta^2 - 2\beta - 1\right)} \qquad (8)$$

To find the correct value of β, one proceeds as in Case I and obtains

$$\beta = \frac{1}{2}\left[\sqrt[3]{\frac{3}{2}\,\frac{A_{slab}}{A'_{col.}} - \frac{1}{2}} - 1\right] \tag{9}$$

A comparison with Eq. 5 indicates that, for equal ratios of slab area to column area, $\beta = \frac{1}{2}(\rho - 1)$ when $r = \frac{1}{2}s$, i.e. when the circular column is inscribed inside the square one. Pursuing this type of comparison further, it can be shown that Eq. 8 and 9 give a considerably higher punching load for any given set of physical parameters, than does use of Eq. 4 and 5 with $r = \frac{s}{\sqrt{2}}$, i.e., for the circumscribed circular column. When $r = \frac{1}{2}s$, of course, the difference will be even greater. For geometrical admissibility the yield line pattern perimeter must be parallel to the column perimeter. The above analysis therefore indicates that when a square column punches through a slab in a bending failure, the yield line pattern should be that of a circular column. Furthermore, since the lowest punching strength will be obtained with the smallest column radius, one would predict that the punching strength of the square column would lie somewhere between that of the circumscribed and that of the inscribed circular column. For the sake of safety, one should assume that of the inscribed circular column in design. Experimental results have shown formation of circular yield line patterns around square column stubs and have been discussed previously.[4]

Case III–Exterior, Circular Column Bisected by the Slab Boundary

The column of Case I has been moved to the edge of the slab so that the edge coincides with the diameter of the column (see Fig. 8-6). In this case, the shear along the perimeter of the fan will be

$$q = \frac{P - \frac{1}{2}\pi R^2 w}{\pi R} \tag{10}$$

and the virtual work equation will be identical to Eq. 3, except that the integration will now be from 0 to π. The solution then becomes

$$\frac{P}{m(1 + k_m)} = \frac{wA_{slab}}{m(1 + k_m)} = \frac{6\pi\rho\,\dfrac{A_{slab}}{A_{col.}}}{6(\rho - 1)\,\dfrac{A_{slab}}{A_{col.}} - (\rho^3 - 1)} \tag{11}$$

with

$$\rho = \sqrt[3]{3\,\frac{A_{slab}}{A_{col.}} - \frac{1}{2}} \tag{11a}$$

The moment $M_{col.}$ which can be produced in the column by the slab can be found by integrating the negative slab moment m around the half perimeter of the column and adding the moment caused by the shear acting on the half column periphery:

$$M_{col.} = 2mr + w\left(A_{slab} - \frac{1}{2}A_{col.}\right)\frac{2r}{\pi} \tag{12}$$

Eq. 11 and 11a are shown in graphical form in Fig. 8-4.

Case IV–Circular Column near the Slab Boundary

The column of Case I is so located that its center is a distance $a \geqslant r$ from a straight slab boundary (see Fig. 8-7a). If the slab outside the yield fan remains plane and horizontal, as it must in small deflection theory and real structures, geometrical and continuity considerations require that the yield fan perimeter be parallel to the column perimeter, i.e. a circular arc. The radius of this arc will again be called R. The distance from the center of the column to any point on the slab boundary between its two intersections with the yield fan arc will be called R_1. Obviously, $R_1 = a/\cos\phi$, where ϕ is measured as shown in Fig. 8-7a. As a further aid to calculations, let ϕ_1 be the value of ϕ for which $R = R_1$.

It is obvious that the yield fan segments which do not touch the slab boundary will be identical to those of Case I, and that, therefore, their part of the formulation of the virtual energy equilibrium equations will offer no difficulty. The fan segments which *are* bounded by the edge of the slab, however, must be examined further. These segments are irregular quadrilaterals, which would be very difficult to analyze. Two possible simplifications are shown in Fig. 8-7b and 8-7c. Use of the latter would cause loss of the effect of the small triangle at the tip, since its area is a function of $(d\phi)^2$, while all other areas, and the integration, will only deal with the first power of the differential. In the model of Fig. 8-7b, on the other hand, the loss of the small tip triangle is offset by the adjacent added triangular area. This model of the segment was therefore used in the formulation. Then, letting the rim of the fan drop a unit distance, one can write the virtual work equation:

$$\int_{\phi_1}^{\pi} qR d\phi + w\left[\frac{1}{2}(R - r)(d\phi)(R - r)\frac{2}{3} + r d\phi (R - r)\frac{1}{2}\right] +$$

$$\int_{0}^{\phi_1} w\left[\frac{1}{2}(R_1 d\phi - r d\phi)(R_1 - r)\frac{2}{3}\frac{R_1 - r}{R - r} + r d\phi (R_1 - r)\frac{1}{2}\frac{R_1 - r}{R - r}\right] =$$

$$\int_{\phi_1}^{\pi} m(1 + k_m)\frac{R d\phi}{R - r} + \int_{0}^{\phi_1} m \frac{R_1}{R - r}\, d\phi \tag{14}$$

In this case,

$$q = \frac{w(A_{slab} - A_{fan})}{2R(\pi - \phi_1)} \tag{15}$$

Letting $\rho = \frac{R}{r}$ and $a = \frac{a}{r}$, one can express the trigonometric functions of ϕ_1 in terms of ρ and a. Therefore, integrating Eq. 14 and making the indicated substitutions, one obtains:

$$\frac{P}{m(1 + k_m)} = \frac{wA_{slab}}{m(1 + k_m)} =$$

$$6\pi \frac{A_{slab}}{A_{col.}} \left\{ \frac{a}{1 + k_m} \log_e \left(\frac{\rho}{a} + \sqrt{\frac{\rho^2}{a^2} - 1} \right) + \rho\left(\pi - \text{arc cos} \frac{a}{\rho}\right) \right\} x$$

$$\left\{ a^3 \left[-2 \frac{\rho}{a} \sqrt{\frac{\rho^2}{a^2} - 1} + \log_e \left(\frac{\rho}{a} + \sqrt{\frac{\rho^2}{a^2} - 1} \right) \right] + \text{arc cos} \frac{a}{\rho} + \right.$$

$$\left. 3\pi(\rho - 1) \frac{A_{slab}}{A_{col.}} \cdot (\rho^3 - 1)\left(\pi - \text{arc cos} \frac{a}{\rho}\right) \right\}^{-1} \tag{16}$$

This expression again contains ρ as an undetermined quantity. In this case, however, it will be difficult, if not impossible, to evaluate ρ directly. Instead, it is more practical to use an iteration procedure with a series of trial values of ρ to obtain the least value of $\frac{P}{m(1 + k_m)}$ for each set of the physical parameters. The results of a large number of such calculations are presented in Fig. 8-9, which shows the punching capacities, and Fig. 8-11, which gives magnitudes of the fan radii. Fig. 8-9 also shows two other failure modes which may govern in this Case. When a is relatively small, and $A_{slab}/A_{col.}$ is relatively large, the mode pictured in Fig. 8-7 and expressed by Eq. 16 will give the lowest punching strength. When a is relatively large and $A_{slab}/A_{col.}$ is small, a full circular fan will form, and the interior column punching expression, Eq. 4, will govern. Finally, if several columns are placed in a line, parallel to the edge of the slab and some distance from it, a cantilever failure may occur along a line tangent to the column perimeters. The equation of cantilever bending equilibrium applicable to this failure mode can be expressed in the form:

$$\frac{P}{m(1 + k_m)} = \frac{2\pi \frac{A_{slab}}{A_{col.}}}{(1 + k_m)(a - 1)^2} \tag{17}$$

This equation is represented by the dashed lines in Fig. 8-9.

In this connection one can also examine the problem of a single circular column supporting an isolated, square slab at its center, (which is identical to a square footing supporting a circular column). In that case, one can rewrite Eq. 17 to obtain

$$\frac{P}{m(1+k_m)} = \frac{2\pi \dfrac{A_{slab}}{A_{col.}}}{\left(\sqrt{\dfrac{A_{slab}}{A_{col.}}\dfrac{\pi}{4}} - 1\right)^2 (1+k_m)} \tag{18}$$

If one compares this with Eq. 4 and 5, one finds that cantilever failure will occur in the square slab when k_m is greater than a number which ranges from 0.16 to 0.25 for the magnitudes of $A_{slab}/A_{col.}$ considered herein. Since such a slab or footing is likely to be reinforced on only one side, the condition required for a circular fan yield line failure will normally be met.

Case V–Circular Column near a Corner of the Slab

The column of Case I is so located that its center is a distance $a \geqslant r$ from each of two slab boundaries meeting in a 90 deg corner (see Fig. 8-8). The virtual work equation can be formulated in the same manner as that for the column near a straight boundary, except that the limits of integration and the expression for q will be slightly different. It should also be noted that, when $a/r = 1$, the fan will not extend into the region between the column perimeter and the corner. Thus two different expressions will be obtained:

When $a > 1$

$$\frac{P}{m(1+k_m)} = \frac{wA_{slab}}{m(1+k_m)} =$$

$$6\pi \frac{A_{slab}}{A_{col.}} \left\{ \frac{a}{1+k_m} \left[\log_e\left(\frac{\rho}{a} + \sqrt{\frac{\rho^2}{a^2} - 1}\right) + 0.88137358 \right] + \rho\left(\frac{3}{4}\pi - \text{arc cos}\,\frac{a}{\rho}\right) \right\} \times$$

$$\left\{ a^3 \left[-2\frac{\rho}{a}\sqrt{\frac{\rho^2}{a^2} - 1} + \log_e\left(\frac{\rho}{a} + \sqrt{\frac{\rho^2}{a^2} - 1}\right) - 3\frac{\rho}{a} + 2.295587148 \right] + \right.$$

$$\left. \left(\frac{\pi}{4} + \text{arc cos}\,\frac{a}{\rho}\right) + 3\pi(\rho - 1)\frac{A_{slab}}{A_{col.}} \cdot (\rho^3 - 1)\left(\frac{3}{4}\pi - \text{arc cos}\,\frac{a}{\rho}\right) \right\}^{-1} \tag{19}$$

When $\alpha = 1$

$$\frac{P}{m(1 + k_m)} = \frac{wA_{slab}}{m(1 + k_m)} =$$

$$6\pi \frac{A_{slab}}{A_{col.}} \left\{ \frac{1}{1 + k_m} \log_e \left(\rho + \sqrt{\rho^2 - 1}\right) + \rho\left(\frac{3}{4}\pi - \text{arc cos}\frac{1}{\rho}\right) \right\} \times$$

$$\left\{ -2\rho\sqrt{\rho^2 - 1} + \log_e\left(\rho + \sqrt{\rho^2 - 1}\right) + 3(\rho - 1)\left(\pi \frac{A_{slab}}{A_{col.}} - 1\right) + \text{arc cos}\frac{1}{\rho} - \right.$$

$$\left. (\rho^3 - 1)\left(\frac{3}{4}\pi - \text{arc cos}\frac{1}{\rho}\right) \right\}^{-1} \tag{20}$$

ρ is again an undetermined quantity in these equations and the correct punching load is best found by iteration for minimum $\frac{P}{m(1 + k_m)}$.

In the case of corner columns it is not only possible, but certain, that, for some values of the parameters, the corner will break off beyond the column. See also Fig. 8-8. The cantilever bending equilibrium equation for this type of failure can be written as:

$$\frac{P}{m(1 + k_m)} = \frac{6\pi \frac{A_{slab}}{A_{col.}}}{(1 + k_m)(\alpha\sqrt{2} - 1)^2} \tag{21}$$

The results of calculations with Eq. 19, 20, and 21 are plotted in Fig. 8-10, while some associated values of ρ are shown in Fig. 8-11. In Fig. 8-10, the straight lines ascending from left to right were obtained by use of Eq. 21, while the curved lines descending from left to right were given by Eq. 19 and 20. As can be seen, the interior column punching strength governs for only a few cases and only when $k_m = 0.5$. It is apparent from Fig. 8-11, that the fan radius for minimum punching strength is not very sensitive to the value of α. Furthermore, ρ changed by less than 5 percent when k_m was varied from 0.5 to 2.0. It was therefore decided to only present the values of ρ for $k_m = 1$ here.

LOWER BOUND AND SHEAR ANALYSIS

Due to the lack of experimental data, it was felt to be desirable to check the upper bound yield line analysis with elastic analysis, and also to compare the predicted strengths with those obtained from an ACI Code (ACI 318-63 and ACI 318-71) shear analysis. The elastic analysis was carried out by finite difference solution of the partial differential equation for elastic slabs. It was supplemented by an incremental analysis in which yield hinges were introduced at the finite difference mesh nodes at which the principal moment reached the Code ultimate, as the load was incremented in steps. The hinges were created by setting the appropriate node stiffnesses to very low values for all load increments above the one first causing yield moment at the node. The model used was a square slab supported on 4 square columns at the mid-points of the sides. The mesh size was 1/32 times the side length of the slab, and the column side length was 1/16 that of the slab. This gave $A_{slab}/A_{col.} = 64$. k_m, of course, was = 1. Then for $a = 1$, the elastic theory gave $\frac{P}{m(1 + k_m)} = 1.65$, while for $a = 2$ it gave 1.96. The incremental method gave $\frac{P}{m(1 + k_m)} = 5.49$ for $a = 1$ and 6.12 for $a = 2$. From Fig. 8-9 the yield line theory values would be 5.4 and 6.4 respectively. Some inaccuracy is inevitable in the incremental method, since the mesh size used was rather large compared to the column size and since, furthermore, only 6 loading increments were used with the first model and 7 with the second one. This meant that several node points had to have their stiffnesses reduced to very low values in the same increment, even though their principal moments did not all precisely reach the Code ultimate. The principal moment directions were also recorded, and they indicated that typical yield fans were in the process of forming as the loads increased.

To check against the ACI Code shear analysis, it is convenient to rewrite the Code formulas for an interior circular column as follows:

$$4\phi\sqrt{f'_c}\,\pi\,(2r + d)d = P - w\pi\left(r + \frac{d}{2}\right)^2 \tag{22}$$

where d is the effective depth of the slab and ϕ is the Code capacity reduction factor. Then, letting $7/8\ d$ be the distance between the centroid of tension and the centroid of compression in the slab in bending, one can write the yield moment as:

$$m = \frac{7}{8} p\ f_y d^2\ \phi \tag{23}$$

Taking ϕ as having the same value for both shear and bending, which is a small approximation, dividing Eq. 22 by Eq. 23, letting $d/r = \delta$ and performing some algebra, one obtains:

$$\frac{P}{m} = \frac{28.7\ \frac{A_{slab}}{A_{col.}}\left(1 + \frac{1}{2}\delta\right)\sqrt{f'_c}}{\left[\frac{A_{slab}}{A_{col.}} - \left(1 + \frac{1}{2}\,\delta\right)^2\right] p\delta f_y} \tag{24}$$

Fig. 8-12 expresses Eq. 24 in graphical form. For columns near enough a slab boundary so that $a < r + ½d$, P must be multiplied by a reduction factor approximately equal to $(\pi - \theta)/\pi$ where $\theta = \arccos\,[a/(1 + ½\delta)]$. (The vicinity of the boundary will also have an effect on P when $a > r + ½d$, but this effect is not susceptible of direct shear analysis since it will be a bending phenomenon.)

A comparison of Fig. 8-9 and 8-12 will show that, for large columns near a boundary of a thin slab, even the very conservative ACI Code shear formula predicts a greater punching strength than yield line theory indicates is present. Actually, of course, it was shown previously[4] that slabs at interior columns are apt to be much more resistant to punching shear than is predicted by the ACI Code. Lacking experimental data, one must be cautious about extrapolating this finding to columns near a boundary or corner. However, it should be noted that the constant Q contains the column perimeter in the denominator. If one then postulates a decrease in the *effective* column perimeter due to proximity of the boundary, one can obtain an increase in the apparent value of Q, which will be a warning of possible shear instead of bending failure. The presence of shear reinforcement, of course, would greatly increase the permissible value of P in Eq. 22 and 24, and would therefore increase the likelihood of yield line bending failure.

EXAMPLES AND COMPARISONS

The use of the foregoing theory will now be illustrated by means of a design example. A flat plate floor is to be designed for a service load of 100 lb per sq. ft. The column spacing is to be 25 ft center to center in each direction, and the columns will be circular, with a 15 in. diameter. Slab thickness will be assumed to be 10 in., as determined by deflection requirements, the yield strength of the reinforcement will be set at 60 ksi and the concrete strength will be set at f'_c = 4000 psi. The concrete weight will be taken as 145 lb per cu ft.

For ultimate strength analysis, the loading then will be:

$$w_{ult.} = \frac{10}{12}(145)(1.5) + (100)(1.8) = 361 \text{ psf}$$

Considering an interior column first,

$$\frac{A_{slab}}{A_{col.}} = \frac{625}{1.25^2\left(\frac{\pi}{4}\right)} = 510$$

Then, from Fig. 8-4, one can find that $\frac{P}{m(1 + k_m)} = 7.5$. Since $P = (625)(0.361) = 226$ kips, it is obvious that $m(1 + k_m) = 226/7.5$ in.-kips/in. In the vicinity of an interior column it would seem appropriate to distribute the reinforcement so as to make the positive moment resistance of the slab half the negative moment resistance, i.e., $k_m = ½$. Thus one finds that the negative moment capacity of the slab must be $m = 20$ in.-kips/in., in all directions. From Eq. 23, with $\phi = 0.9$ and average $d = 8½$ in., one can find that $p = 0.0059$. This is the negative reinforcement ratio required to prevent yield line punching failure under the given conditions and loads. The positive reinforcement ratio, of course, will be one half that, and must also be provided.

It is now necessary to check this design against a punching shear failure. As discussed earlier and also in Reference 4, one can be assured that the punching failure will be of the bending (yield line) type, rather than in shear, if the parameter $Q < 2$. Eq. 1 gives the expression for this parameter. It will be assumed here that the positive moment reinforcement will be detailed to extend into the columns, thus providing some dowel resistance to punching shear. Then, according to Reference 4, the value of p to be used in Eq. 1 is that for the negative moment reinforcement alone. Substituting the appropriate values (note that $b = 15\pi$ in. and $B = 1200$ in.) into the expression for Q, one obtains:

$$Q = \frac{p^2 f_y d^2}{\sqrt{f'_c} bB} = \frac{(0.0059)^2\,(60{,}000)\,(8.5)^2}{\sqrt{4{,}000}\,(\pi)\,(15)\,(1200)}\,(10)^4 = 0.42 < 2$$

Thus there is no danger of a shear failure around these columns at the given loads.

It should be noted that this analysis does not guard against other kinds of failure in the slab as a whole. It only assures that, given the column configurations and slab parameters assumed here, no punching failure of the column through the slab will occur under the given load. It is quite likely that additional reinforcement will be required to prevent failure of the slab system in modes other than the punching one. This reinforcement, obviously, will not reduce the punching resistance, though it may increase Q to a point where shear failure could occur before bending failure, though at a considerably higher load than that for which the slab/column intersection was originally designed.

If the same column were located tangent to an exterior boundary of the same flat plate floor, i.e., with $\alpha = 1$, one could assume that the effective $A_{slab}/A_{col.}$ would be one half that of the interior column, which would make it equal to 255. At the same time, one might make the positive and negative moment reinforcement equal at that location, so that $k_m = 1$. Then, from Fig. 8-9, one finds that $\frac{P}{m(1 + k_m)} = 4.4$ for this case. From this, $m = \frac{113}{(4.4)(2)} = 12.8$ in.-kips/in. The negative reinforcement requirement then becomes $p = 0.00375$ from Eq. 23.

Checking for shear failure, one would have to change the values of b and B from those used for the interior column. For a column so close to the boundary, one could conservatively use half the perimeter as resisting shear, so that $b = \frac{1}{2}(\pi)(15)$ in., while B will now become 900 in. Then, provided the positive moment reinforcement is again carried into the column, $Q = 0.45$. If this is not done, the value of p in Eq. 1 must be taken as the sum of the reinforcement ratios for positive and negative bending, which would make $Q = 1.80$. Accordingly, shear should not be critical for this case either.

A column in a corner of the slab can be handled similarly, except that Fig. 8-10 must be used in place of Fig. 8-9, and that the values of P, b and B must be reduced appropriately.

It will be instructive to analyse the interior slab/column intersection designed above, according to the ACI Code requirements for shear and also by some other proposed punching analyses. Only the interior column intersection can be so checked, since neither the ACI Code nor the other proposed formulas are directly applicable to unsymmetrically loaded slab/column intersections.

Fig. 8-12 is useful in carrying out the ACI Code analysis. For the example discussed above, $\delta = 8.5/7.5 = 1.13$. Then, from Fig. 8-12,

$$\frac{P}{m}\left(\frac{pf_y}{\sqrt{f'_c}}\right) \doteq 42$$

Solving for P with $m = 20$ in.-kips/in., $p = 0.0059$ and the physical parameters given previously, one finds that $P = 150$ kips. This, and the other comparative values for the given intersection, are listed in Table I below:

TABLE I—COMPARATIVE VALUES OF PUNCHING LOAD (KIPS)

Yield Line Punching	ACI Code	Moe[5]	Whitney[6]	Yitzhaki[7]	Mowrer and Vanderbilt*
226	150	208	109	205	266

*For lightweight concrete.

SUMMARY AND CONCLUSIONS

The slab-column interaction of uniformly loaded flat plate structures has been examined by use of Yield Line Theory. Appropriate expressions have been derived for the bending-punching strength of such plates, when supported on columns located in the interior, or at or near the boundaries. Comparison of the expressions for the former with experimental data available in the literature, has shown that the bending-punching phenomenon will frequently be the governing one for interior columns. Comparison of the expressions for the latter with the ACI Code punching *shear* analysis, (in the absence of experimental data) indicates that bending-punching must also be considered as a possible failure mode for columns at or near a boundary or corner. This will particularly be the case when shear reinforcement is used around the columns. Thus it appears that the single most important conclusion that one can draw from this study is that bending-punching failure is possible around all columns and that yield line theory can be used to analyze slabs for it.

The yield line theory expressions derived herein are mainly applicable to circular columns. However, it was shown that square columns may also be expected to cause bending-punching failure in a circular mode, and that, therefore, the bending-punching strength of a slab resting on square columns can be predicted from the expressions for circular columns.

Comparison of yield line theory with some elastic, lower bound solutions, indicates that the latter severely underestimate the bending-punching strength.