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Stability of Reinforced Concrete Shells: State-of-the-Art Overview

By Egor P. Popov and Stefan J. Medwadowski

Synopsis: Over the last few decades, shell structures have become bigger--they cover larger areas without intermediate supports--and thinner. Because of this, the problem of buckling of shells has grown in importance. This paper contains an overview of the general problem of stability of reinforced concrete shells. The buckling phenomenon is defined and its manifestations in columns, plates, and shells are discussed. The linear critical load concept is reviewed first, followed by a consideration of geometric nonlinearities and of geometric imperfections of the shape of the shell as built. Next, the material properties of reinforced concrete and its response under load are reviewed. The properties of inelastic behavior of concrete and reinforcement, the cracking of concrete, the amount of reinforcement, and the effects of concrete shrinkage and creep are discussed. These factors make the buckling behavior of reinforced concrete shells significantly different from metallic shells and cause a reduction in the load-carrying capacity of the shell. Current approaches to shell stability analysis and design are commented on.

Keywords: buckling; cracking (fracturing); creep properties; loads (forces); reinforced concrete; shells (structural forms); shrinkage; stability; structural analysis; structural design.

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INTRODUCTION

Domical forms have been used by man from time immemorial, for shelters, utilitarian and symbolic structures. Early materials were natural wood, saplings, hides, stone, clay. Methods of construction were largely a matter of tradition, strongly influenced by local conditions and materials. Even though progress was dispersed and slow, evidence of some impressive structures has survived to our own time.

Decisive progress was achieved in Rome; the most prominent example—partly because of its size, partly because of its beauty, and partly because of its having survived to our own time—is the Roman Pantheon, constructed approximately 1850 years ago. The structure consists of a hemispherical dome supported on a cylindrical drum. The diameter of the dome is approximately 43.3 m, resulting in a most impressive space, made possible by the use of the native pozzolean cement in the construction—in effect, the Pantheon is constructed of concrete. The dome is ribbed to lighten the dead load, as is the drum of the walls. At the edge of the large lantern opening at the top of the dome, the shell is approximately 1.2 m thick; the thickness increases toward the springing line. The total span of the dome was not exceeded until modern times (cf. TABLE 1).

The tradition of domical forms continued, first in Roman architecture, in Byzantium (with the magnificent example of the Church of St. Sophia in Constantinople), later in the Romanesque, the Gothic and the Renaissance, with many buildings remaining as testimony to the daring and skill of the builders. Domes on a noncircular plan, domes with pendantives, ribbed and skeletal shells were added to the tradition. The use of concrete, however, so common in ancient Rome, gradually diminished and it became the forgotten material, to be rediscovered at the time of the Industrial Revolution[22].

The Industrial Revolution witnessed many changes in the manner in which man built his structures. A rapid development of what is now called structural mechanics brought with it rational methods for the analysis of shells, the first comprehensive theory having been published by Aron in 1874[10], to be followed soon by the seminal writings of Love[72]. At the same time, new materials of construction were developed: cast iron, later to become steel, and reinforced concrete, a material of particular importance because of its moldability and the ease with which it could be formed into surface structures. Not surprisingly, the first reinforced concrete shell structures appeared before 1918. Examples are the Orly airport hangar by Freyssinet, a corrugated cylinder with only a limited spatial action, but a most impressive project, nevertheless, and some barrel vaults by Perret[22], delicate and well proportioned, and yet intensely practical at the same time, offering a promise of things to come.

Rapid growth in the construction of reinforced concrete thin shells came in the decade of the twenties. The rather forbidding formulations of the theory of thin shells were simplified, first for spherical caps by Geckeler[36], then by Finsterwalder[33] for circular cylindrical barrel vaults; in each case, the motive was the direct need of the construction industry. New forms appeared. Elliptic paraboloids were experimented with, Freyssinet[57] used conoidal north-light shells for railway sheds, where he achieved structures of great beauty, and finally the hyperbolic paraboloid appeared, first in the realized projects of Baroni, later in the theoretical investigations of Aimond and Lafaille, to re-emerge triumphant in the exuberant architecture of Candela[32]. At the same time, new construction techniques appeared: guniting and rigid reinforcement (similar to the Melan system of bridge construction, but extended to three-dimensional surface structures) for the Jena Planetarium, precast skeletal shells in the many projects of Nervi[80], use of cable reinforcement for shells, and finally the use of prestressing.

TABLE 1 COMPARISON OF SHELL DIMENSIONS AND THICKNESS-TO RADIUS-OF CURVATURE RATIOS

Project	Location and Year	Type of Geometry	Plan dimensions	Radius of curvature R	Thickness h	$\approx h:R$
The Pantheon	Rome *ca.* 100	hemisphere	43.3 m (dia)	21.65 m	1.2m at top	1:24
Planetarium	Jena 1923	hemisphere	25 m (dia)	12.5 m	.06m	1:200
Factory	Jena 1923	spherical cap	40 m (dia)	28.28 m	.06 m	1:470
Market Hall	Algeciras 1934	spherical cap on 8 supports	47.6 m (dia)	44.1 m	.09 m	1:490
Filter Plant	Minnesota 1939	ellipsoid of revolution	45.7 m (dia)	47.24 to 5.33 m	.09 m to .15 m	1:525 to 1:35
Factory	Bryn Mawr 1947	elliptic paraboloid on rect. plan	19.6 m x 25.3 m	25.0 m to 32.9 m	.09 m	1:300 to 1:400
Auditorium	MIT Cambridge 1955	segment of a sphere on 3 points	48.0 m bet. support points	34.0 m	.065 m	1:520
Shopping Center	Kaneohe, Oahu 1957	groined vault (intersection of 2 tori) or 4 points	39.0 m x 39.0 m bet. support points	39.0 m to 78.0 m	.076 m to .178 m	1:1000 to 1:500
Palazetto di Sport[1]	Rome 1957	spherical cap	58.5 m (dia)	30.9 m	.335 m rib	1:92
CNIT[2]	Paris 1957	groined vault (intersection of 3 cylinders) on 3 points	219 m bet. support points	89.9 m to 420.0 m	1.91 m to 2.74 m	1:47 to 1:153
Hen's egg		surface of revolution		2 cm (min)	.2 mm to .4 mm	1:100 to 1:50

1. precast, ribbed construction; material $h:R$ ratio $\approx$ 1:250

2. 2-layer shell, flange thickness from .06 m to .12 m; material $h:R$ ratio 1:1750

It is important to note that the anticlastic surfaces such as the hyperbolic paraboloid, whatever the method of construction, are the only new structural form of the modern era[76]. The initial impact for their use came from the need for efficient and cheap methods of covering large areas without having to construct huge scaffolding structures characteristic of the traditional methods of erection of reinforced concrete shells.

Although the span of the Pantheon is very great, modern shell structures achieve much larger column-free areas and, what is even more significant, the thickness of the modern shell is orders of magnitude smaller than the thickness of the traditional domes. The trend toward greater spans and thinner shells can be clearly discerned (cf. TABLE 1). To an analyst, the significance of this ever larger radius of curvature-to-shell thickness ratio lies in the realization that it has an important effect on the buckling of concrete thin shells, and on their strength reserve. The thickness of shells, in the case of small spans, is dictated by construction considerations: the need to accommodate reinforcement, the practicalities of placing concrete. Thus, a concrete thin shell of small span possesses an enormous reserve of strength. This is no longer the case if the spans become large.

Early modern designers realized fully the need to protect shells from the possibility of buckling. However, their ability to predict the buckling load was limited by the incomplete state of understanding of the phenomenon and by the practical difficulties, analytical and computational, of performing the necessary calculations. At this stage in the development of the theory, two shell forms were studied at length: the sphere and the cylinder. At first, familiar with the success of the linear buckling theory of Euler in predicting the buckling load of columns, analysts formulated the shell problem in the same linear manner. However, it soon appeared that the critical buckling load obtained in this way did not agree with the values of critical buckling loads obtained for the same shell forms experimentally, the experimental values being many times smaller than the critical load predicted by the analysis.

The efforts to explain the lack of correlation between theory and experiment continued over several decades, and have met with success. It appears that the problem has its root in the fact that a shell is a spatial, surface-type structure. Accordingly, the in-plane state of stress in a differential element of a shell consists not of just one axial stress (as is the case in a column), but of two axial stresses and an in-plane shear stress. If a shell is subjected to an axial compression in one direction, the associated axial stress in the orthogonal direction may be compressive or tensile. If it is tensile, the orthogonal axial stress tends to stiffen the shell, and to reduce the deflections normal to the surface. However, if the orthogonal stress is compressive, it tends to increase the normal deflections, thus weakening the shell and lowering the critical buckling load. The linear shell theory is incapable of predicting this behavior, and recourse has to be made to the nonlinear theory of shells. This presents formidable computational difficulties, in general intractable until the advent of the computer.

An additional complication in attempting to predict the buckling behavior of reinforced concrete thin shells lies in the nature of their material. The cracking of concrete, nonhomogeneity of the material, its creep characteristics, its inelastic properties, all have to be taken into account when studying nonlinear phenomena in shells. Finally, the inevitable imperfections in the geometry of the shell as built compound the difficulties of the problem and reduce the buckling load. These problems, too, are not simple, and can be handled generally only with the aid of computers. At this stage in the development of the theory, only partial success has been attained, but it seems fair to say that the essence of the problem has been captured.

Until recently, the lack of computational facilities precluded either nonlinear analysis, or a detailed consideration of the material or geometric factors. Yet shells were being built, and safely. The designers, guided by their judgment as much as by some approximate analyses, resorted to a number of devices aimed at making reinforced concrete thin shells safe in buckling. These devices included:

- Modification of the shape of the shell. This modification could take several forms. To reduce the radius of curvature of the shell in the small, local undulations might be introduced. Alternatively, the shape of the shell might be changed locally—usually along the edges—so that the radius of curvature might become smaller along a strip of the shell; this device consisted essentially of introducing an edge beam and thus stiffening the edge of the shell. An improvement in the buckling behavior of a shell could be obtained through the use of shells of double rather than single curvature.
- Modification in the manner in which the material is distributed throughout the surface of the shell. This can be done by providing a system of ribs throughout the surface of the shell, in one or two directions (usually orthogonal), as might be required in a given case. Alternatively, all of the material might be placed along the lines of the ribs, as in skeletal (or reticulated) shells, thus providing an "effective thickness" of the shell greater than if the material were evenly distributed throughout the surface of the shell. This type of modification is evident in the projects of Nervi[80] and Torroja[105].
- A variant in the modification of the manner in which the shell material is distributed was used by Esquillan, one of the great pioneer shell designers of the modern era, in the design of the great hall of the CNIT near Paris[30]. This shell has the form of a groined vault on three supports, formed from the intersection of three parabolic cylinder segments, with the span between the three supports of approximately 219 m—the largest shell ever built. The shell itself consists of two thin surface layers, connected by shear-transferring diaphragms. Thus, the total thickness of the shell varies from 1.9 to 2.75 m, while the average thickness of the material is on the order of only 17 cm to 24 cm.
- Finally, the introduction of appropriate prestressing loads may result in an improved buckling behavior. Possibly the best early example of the use of such devices is the famed Market Hall at Algeciras designed by Torroja[105].

The excellence of judgment of shell designers is evidenced by the fact that, as far as we know, very few, if any, concrete shell failures have been conclusively shown to be due to buckling—although some failures said to be due to creep of concrete or to the steep temperature gradient may have been, in fact, buckling failures. (The only failure of a large shell conclusively shown to have been due to buckling involved a reticulated spherical dome made of metal[114].) This is an enviable record, a testimony to the excellence of the designers as well as the inherent strength of shell structures. For this record to be preserved, and in view of the growing number of shells being built, of ever-growing spans and ever-diminishing thicknesses, it is necessary to develop a better understanding of shell buckling behavior.

The purpose of this paper is to provide an overview of the problem of buckling of reinforced concrete shells. The nature of the buckling phenomenon is examined, and its different manifestations possible in shell structures are reviewed. The difference in the qualitative behavior of columns and shells is considered, and the need for a study of postbuckling equilibrium states based on the geometrically nonlinear conditions for certain classes of shells is explained. The influence of geometric imperfections, of the inelastic material properties, of reinforcement, and of cracking of concrete is considered. A brief review of the currently available analytical and experimental methods is made, followed by an examination of current provisions for the study of reinforced concrete shell buckling in two recent recommendations: the ACI[2,4] and the IASS[77]. The paper is to serve as an introduction to the detailed studies of these topics undertaken in the companion papers contained in this volume.

THE BUCKLING PHENOMENON

Consider a straight column subjected to a concentric compressive load. Let the load increase gradually, starting from zero. Within some range of the values of the load, the column remains straight—there is no lateral displacement normal to the axis of the column; the only deformation experienced by the column is a small shortening of its axis.

However, at some value of the compressive load, the following event may take place: the column may experience lateral displacement, and this displacement will increase significantly without a correspondingly significant increase in the magnitude of the load. If such lateral displacement does occur, the column is said to have buckled; the phenomenon itself is called buckling, and the value of the compressive load at which buckling occurs is called the critical load.

It should be noted that the phenomenon of buckling is by no means restricted to that described above, that is, to transverse displacement of columns under axial compression. As an example, the same column subjected to a gradually increasing axial load might experience at some load level a sudden twist while remaining straight—a case of torsional buckling. Similarly, a beam supporting a bending load in its own plane might experience sudden twisting deformation—an example of lateral buckling of beams. Both these phenomena are well known and documented, and provisions for investigating the buckling of struts under compressive load, and for investigating lateral buckling of beams, are contained in all building design codes.

It turns out that plate and shell elements—in fact all structural elements subjected to loads such that internal compressive in-plane stresses are present—are also subject to buckling. However, because of the two-dimensional nature of shells (as opposed to the one-dimensional nature of columns and beams), the problem of buckling of shells is very much more complex in its physical manifestations, and very much more complex to investigate.

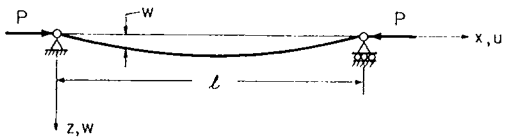

Fig. 1 Simply Supported Buckled Strut

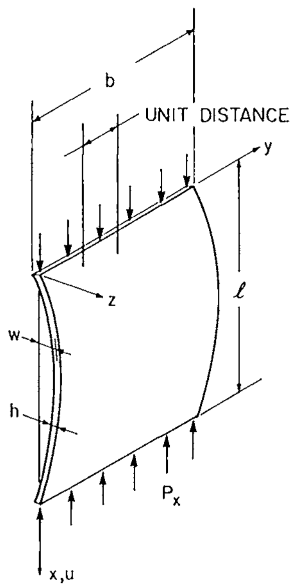

Fig. 2 Buckled Plate Simply Supported at Loaded and Free at Unloaded Edges

PATHS OF EQUILIBRIUM

Introduction

It is of interest to trace the successive positions of equilibrium of structural elements associated with the successive values of slowly increasing loads up to, at, and beyond the critical load. The resulting load *vs.* deflection curve is called the *path of equilibrium.* The *primary path* is the portion of the curve up to the critical load, while the *secondary path* is the portion beyond the critical load. Columns, flat plates, and shells are all of interest because all three elements occur as parts of thin shell structural assemblies. Because of its relative simplicity, the problem of a column will serve as the starting point.

Ideal Elastic Column

Consider again a column under axial load. The column is supposed to be ideally straight, and subjected to ideally concentric compressive load—an ideal column. The length of the column is l, and it is simply supported at the ends. The external compressive load is P.

In order to determine the successive deflected shape of the column in the successive positions of equilibrium—the problem of *elastica*— the assumptions first made by Euler in 1744 are followed[31]; these assumptions have since then been repeated in virtually all standard texts on the strength of materials (see, for example, Ref. 83). We assume that the strain-displacement relations and the constitutive relations of the technical theory of beams apply, as follows

$$\epsilon = \frac{du}{dx} \qquad \kappa = -\frac{d^2w}{dx^2}$$

$$N = AE\,\epsilon \qquad M = EI\,\kappa \tag{1}$$

where ϵ and κ are, respectively, the longitudinal strain and the change of curvature of the column, u and w are its axial and transverse displacements, x is the coordinate (cf. Figure 1), N and M are the axial force (positive if tensile) and the bending moment in the column, A and I are the area and the appropriate moment of inertia of the cross

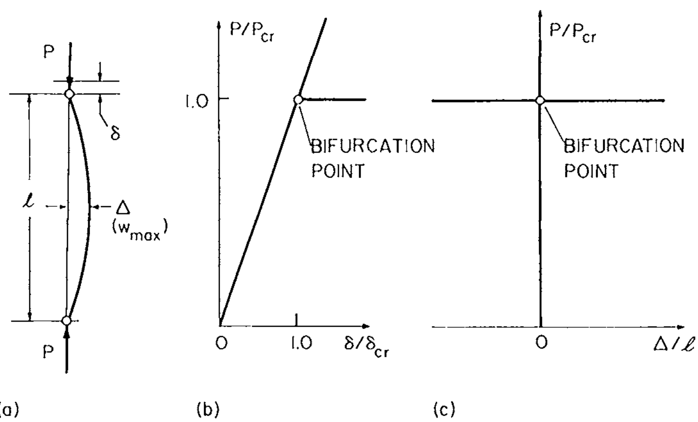

Fig. 3 Buckling Behavior of Ideal Elastic Strut—Linear Theory

section of the column, and E is the modulus of elasticity of the column material. Next, assuming small deflections and negligibly small shear force components in the z-direction, the equations of equilibrium of the deformed differential element of the column can be written:

$$\frac{dN}{dx} = 0$$

$$\frac{dV}{dx} + \frac{dN}{dx}\frac{dw}{dx} + N\frac{d^2w}{dx^2} = 0$$

$$V - \frac{dM}{dx} = 0 \tag{2}$$

Substitution of eq. (2b) into eq. (2c) together with the use of eqs. (1) and (2a), and assuming $EI = constant$, results in

$$\frac{dN}{dx} = 0$$

$$EI\frac{d^4w}{dx^4} - N\frac{d^2w}{dx^2} = 0 \tag{3}$$

This is the system of differential equations that governs the problem, written in the two unknowns N and w. It is nonlinear in the sense that these unknowns appear as a product in the second equation. However, eq. (3a) can be solved for N first, and its known value can then be substituted into eq. (3b). In the case of a compressive force P at the ends, $N = -P$, and

$$EI\frac{d^4w}{dx^4} + P\frac{d^2w}{dx^2} = 0 \tag{4}$$

Consulting Figure 1 it is seen that this equation could be derived also by observing that the moment at any point x of of the column is $M = Pw$ and hence, using eqs. (1),

$$\frac{d^2w}{dx^2} + \frac{P}{EI}w = 0 \tag{5}$$

Successive differentiation of this equation with respect to x, taking $EI = constant$, results in eq. (4). Equation (5) is suited to obtaining solutions of problems involving only static boundary conditions, while eq. (4) can be used to obtain solution of problems involving any boundary conditions of a column.

Nontrivial (*i.e.*, $w \neq 0$) solutions of eq. (4) exist only for certain values—the characteristic values—of the axial force P. The smallest of these values is called the critical value P_{cr}, also called the Euler critical buckling load of the column. The value of the critical load for a pin-ended column is given by the expression

$$P_{cr} = \frac{\pi^2 EI}{l^2} \tag{6}$$

For a column of rectangular cross-sectional area A and the smaller side dimension h, the moment of inertia is $Ah^2/12$, and the critical load can be put in the form:

$$P_{cr} = \frac{\pi^2 EA}{12(l/h)^2} \qquad \sigma_{cr} = \frac{\pi^2 E}{12(l/h)^2} \tag{7}$$

where the quantity $12(l/h)^2$ is termed the *slenderness ratio*, and $\sigma_{cr} = P_{cr}/A$.

Before considering further the solution of eq. (4), and to simplify a comparison with plates and shells, consider a rectangular column of width b (comparable in magnitude to its length l) shown in Figure 2—in effect a plate, with the loaded edges simply supported and the unloaded edges free. Let the total column load be $P = P_x b$, and the flexural stiffness be $EI = Db$, where

$$D = \frac{Eh^3}{12(1-\nu^2)} \tag{8}$$

and where ν is the Poisson ratio of the column material. Then eq. (4) becomes

$$\frac{d^4w}{dx^4} + \frac{P_x}{D}\frac{d^2w}{dx^2} = 0 \tag{9}$$

This equation will be referred to at several points later in this paper.

Solution of eq. (4) can be plotted as is done in Figure 3. As the axial force P increases, axial shortening of the column is observed. However, for the assumed ideal column, no lateral deflection w will occur until the critical load P_{cr} is reached. At that instant, the stable equilibrium configuration of the column is that of a slightly bent element. The point on the path of equilibrium where the magnitude of the applied load is equal to the critical load is called the *bifurcation point,* since at that point the path of equilibrium splits into two branches. The column may take either of two possible equilibrium configurations: the straight one or the slightly bent one. However, the solution of eq. (4) does not provide any information about the magnitude of the lateral displacement Δ at the midheight of the column, nor about the slope and the shape of the branched path of equilibrium, respectively at and beyond the bifurcation point. The lateral displacement Δ can take on any value, subject only to the requirement that it remain small within the meaning of the assumptions made in the derivation of eq. (4). These assumptions are evidently too restrictive to allow the path of equilibrium corresponding to the bent configuration of the column to be plotted beyond the bifurcation point.

A complete solution of the problem was first obtained by Lagrange[68]. Although his formulation was somewhat different (cf. Ref. 102), it was based on exact expressions for strain and curvature, instead of the approximate relation given in eq. (1). The plot of the corresponding path of equilibrium is shown in Figure 4. It is seen that after the bifurcation point has been reached, the column, while in the bent position, can support a load larger than the critical load; the critical load itself is the same as the critical load predicted by the linear theory. However, the lateral deflections become very large, indeed so large as to be inadmissible from the point of view of using the column as a structural element.

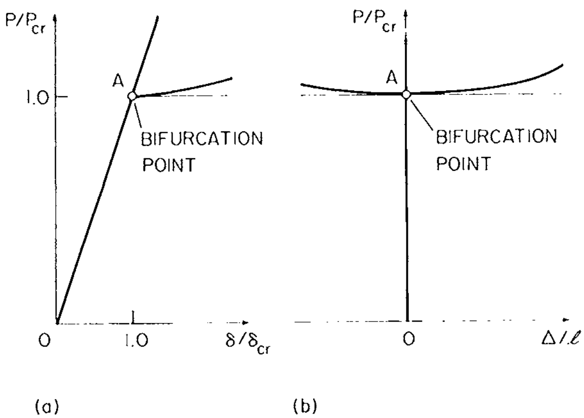

Fig. 4 Buckling Behavior of Ideal Elastic Strut—Nonlinear Theory

The observation that the critical load obtained from the exact solution is the same as the linear Euler critical load leads to the speculation that the nonlinearity included in the analysis need not be strong—approximate nonlinear strain-displacement expressions might suffice if one is interested only in the segment of the path of equilibrium in the immediate neighborhood of the bifurcation point. This turns out to be the case, and the simplest nonlinear theory based on small displacements and moderately large rotations leads to the following approximate expression for the longitudinal strain

$$\epsilon = \frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 \tag{10}$$

This relation replaces the equivalent expression in eqs. (1). The practical importance of eq. (10) in the case of columns is clearly limited, since the exact solution due to Lagrange is available, as has been mentioned. However, it is of great importance in the equivalent problem of plates and shells, as will be discussed later in the paper.

Imperfect Elastic Column

Since ideally straight concentrically compressed columns are unattainable in practice, it is important to investigate the effect of geometric imperfections on the shape of the path of equilibrium. Some aspects of the behavior of imperfect elastic columns are illustrated in Figure 5. The column is assumed to have an initial midpoint deflection of Δ_0. On gradually increasing the applied axial load, the center deflection increases (Figure 5a). By using a linear theory of the type given by eq. (4), the solution shown qualitatively in Figure 5b is obtained. The paths of equilibrium vary depending on the magnitude of the initial eccentricity. However, regardless of the magnitude of Δ_0, the critical Euler load serves as an asymptote for the solution. As lateral deflections increase, the linear solution (based on small displacements) becomes less reliable. The more accurate solutions for the imperfect elastic column, based on a nonlinear equation more exact than eq. (10), are shown in Figure 5c. The paths of equilibrium suggest that if very large lateral deflections are permitted, the initial eccentricity will play only a very minor role in the final capacity of the column. It must be emphasized again that the resulting deflections are very large. From Figure 5c it is seen that the lateral deflection may be as large as 0.4 of the column length l—this occurs when a pin-ended column is bent into a complete "circle." Such large deflections are completely out of the range of magnitudes admissible in practical applications in structures.

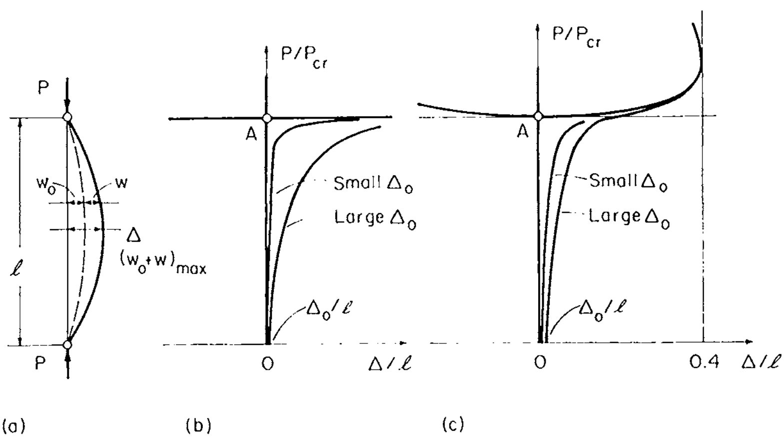

Fig. 5 Buckling Behavior of Initially Bent Struts[107]