Arch Analysis

At each section along the span an "arch slice" is taken as a free body diagram, Fig. 11-9. The arch is in vertical equilibrium under the action of the applied external load and the so called "specific shear," $N'_{xy} = \partial N_{xy}/\partial x$. N'_{xy} is simply the difference between the membrane shears on the two faces of the arch slice and can be calculated at any point by

$$N'_{xy} = \frac{\partial N_{xy}}{\partial x} = \frac{\partial V}{\partial x}\frac{Q}{I} = W\frac{Q}{I} \tag{3}$$

If the longitudinal distribution of the load W is uniform, all arch slices will be loaded identically and thus only a typical slice need be considered. If not, arch slices must be taken at several sections along the longitudinal span.

If a single shell with free edges is being analyzed, the arch is statically determinate, Fig. 11-9b, and the transverse internal forces N_y, M_y, and Q_y can be found directly at any cut section along the arch. If a typical interior shell is being analyzed, the horizontal displacement and rotation at the longitudinal edges must be zero and thus the arch analysis becomes statically indeterminate to the second degree, Fig. 11-9c. Any standard method of indeterminate analysis can be used to find the redundants X_1 and X_2. Chinn[15] has derived formulas based on the elastic center method which simplify this analysis for the common circular cross-section. For very long interior shells, $L/R > 4$, Parme and Connor in discussing Chinn's paper have shown that the effect of membrane displacements in addition to flexural displacements should be considered in the arch analysis to obtain accurate results.

Several points should be recognized regarding the approximations inherent in the analysis. The arch analysis assumes each arch slice is free to deform transversely. Actually, near the diaphragm supports this is not possible so that only at sections some distance from the supports can the internal forces N_y, M_y, and Q_y reach the values found above. The assumption used in the beam analysis was that the cross-section did not distort, while the arch analysis indicates transverse moments exist and thus some distortion of the cross-section does occur. If the distortion due to arch action is small the assumption of a linear distribution for N_x should be good and thus the entire analysis should be quite accurate. It is evident from this that a better approximation is obtained for loads uniformly distributed in the transverse direction rather than concentrated, Fig. 11-8.

Prestressed Shells

A number of simple and continuous prestressed cylindrical shells and *folded* plates have been successfully designed by the beam method.[16,17] The prestressing cables are draped parabolically in the shell surface, Fig. 11-10a, so that in the beam analysis, Fig. 11-10b, the external load is balanced by the upward force exerted by the prestressing cables. Thus only a uniform longitudinal compressive stress N_x is produced and no membrane shear stresses N_{xy} exist. For the arch analysis, Fig. 11-10c, because the specific shear is zero, the vertical applied load on each arch slice is held in equilibrium by the cable forces which have both vertical and horizontal components. Normal reinforcing steel is generally used to resist the transverse moments found from the arch analysis.

Special consideration should be given to local stresses at the end anchorages and also to any possible overload conditions which might produce a considerable magnification of the stresses obtained for a perfectly load balanced design based on a fixed load.

Advantages and Disadvantages of Beam Method

Advantages of this method are:

1. It is based on a simple theory which is universally known.
2. It emphasizes the major structural action involved in many shells, and it can be applied with equal ease to simple and continuous shells and to symmetrical cross-sections of arbitrary shapes.
3. Prestressed shells can be easily treated using the load balancing concept.

Disadvantages of this method are:

1. It is an approximate method which is useful for long shells, and except for certain special cases its exact range of validity is unknown.
2. For unsymmetrical cross-sections or cases where the resultant load does not pass through the shear center of the cross-section, its application becomes extremely complex and questionable, and in these cases it offers no particular advantage over more exact methods.

DIRECT STIFFNESS HARMONIC ANALYSIS

General Remarks

This method was originally developed by Jenkins[6] for the analysis of cylindrical shell systems. De Fries-Skene and Scordelis[18] first utilized the method for the analysis of folded plate systems. A number of additional papers have also used or extended the procedure.[19,20,21,22,23,24]

The problem to be solved is the determination of the internal forces and displacements in a structural system consisting of an assembly of longitudinal shell elements, plate elements and beam elements, Fig. 11-4, interconnected at joints along their longitudinal edges and supported by transverse diaphragms. The known quantities input into the problem include geometry, dimensions and material properties of the structural elements, the surface and joint loadings, and the boundary conditions along the longitudinal joints and at the transverse diaphragms.

Direct Stiffness Method for Simple Span Structures

For a simple span structure, Fig. 11-5a, an analysis for applied loads with any arbitrary longitudinal distribution may be performed using a harmonic analysis. The applied loads are first resolved into Fourier series components. An analysis is carried out for all of the loading components of each particular harmonic and then the final results are obtained by summing the results for all harmonics used to represent the load. Once the solution technique, which involves extensive computations, has been developed for a single harmonic, it can be reused for any harmonic, and thus the method is ideally suited to the application of a digital computer.

The analysis for each harmonic load has the advantage that such loads will produce displacements of the same variation and vice versa and thus a single characteristic value may be used to describe any force or displacement pattern. For example, the displacement pattern:

$$r(x) = r_o \sin \frac{n\pi x}{L} \tag{4}$$

may be described by the single value r_o. This makes it possible to treat an entire joint as a single nodal point and to operate with single forces and displacements instead of functions. If the conditions of static equilibrium and geometric compatibility are maintained at a nodal point they will automatically be satisfied along the entire longitudinal joint. Thus, the two dimensional prismatic shell problem may be treated as essentially a one dimensional problem in the transverse direction. A direct stiffness method applied to such a system results in a structure stiffness matrix which is extremely well conditioned for solution since the non-zero coefficients are all grouped in a narrow band along the main diagonal.

Each joint or nodal point has four degrees of freedom: it can displace vertically and horizontally in a plane parallel to the end diaphragms; it can move longitudinally parallel to the joint; and it can rotate about an axis parallel to the joint. These directions define a "global coordinate system" for displacements or forces at the joint, Fig. 11-12 and 11-13. A "local coordinate system" for displacements and forces is defined for each element in any convenient system desired, Fig. 11-11. For example, for the shell element, Fig. 11-11a, longitudinal, tangential, and radial directions at each edge are selected, while for the beam element, Fig. 11-11c, the principal axis directions of the cross-section are selected.

The direct stiffness method has been described in detail in many publications tions[18,20] and thus will be only briefly outlined here. The procedure used is to first fix all of the longitudinal joints against displacements and determine the internal forces in each plate and shell element due to surface loads only. To this fixed edge solution for surface loads is added a solution for joint loads only. The joint loads R are taken equal and opposite to the sum of fixed edge forces at each joint plus any additional external joint loads acting on the system. The latter solution can be obtained as follows.

(1) Element stiffness matrices k are developed for the elements in their local coordinate system, Fig. 11-11. For the shell and plate elements these are 8 x 8 matrices while for the beam elements they are 4 x 4 matrices.

(2) The element stiffness matrices are transformed to a global coordinate system, Fig. 11-12, using displacement transformation matrices, a.

$$\{\bar{S}\} = [a]^T [k]\,[a]\{\bar{v}\} \tag{5}$$

or

$$\{\bar{S}\} = [\bar{k}]\,[\bar{v}] \tag{5a}$$

for the shell and plate elements the 8 x 8 $\bar{k}$ matrix may be partitioned into four 4 x 4 submatrices as follows.

$$\left\{\begin{array}{c}\bar{S}_i\\ \hline \bar{S}_j\end{array}\right\} = \left[\begin{array}{c|c}\bar{k}_{ii} & \bar{k}_{ij}\\ \hline \bar{k}_{ji} & \bar{k}_{jj}\end{array}\right]\left\{\begin{array}{c}\bar{v}_i\\ \hline \bar{v}_j\end{array}\right\} \tag{6}$$

which relates the forces $\bar{S}$, to the displacements, $\bar{v}$, at the i and j edges of the elements.

For the beam elements, a 4 x 4 matrix, is obtained relating external forces applied to the beam with corresponding displacements.

$$\{\bar{S}\} = [\bar{k}_{ii}]\{\bar{v}_i\} \tag{7}$$

(3) Static equilibrium, Fig. 11-13a; requires that the external joint loads R must equal the sum of the element forces $\bar{S}$ acting on the same joint. For example, consider a shell, plate, and beam element meeting at a common joint, Fig. 11-13a.

$$\{R_i\} = \{\bar{S}_i\}_{shell} + \{\bar{S}_i\}_{plate} + \{\bar{S}_i\}_{beam} \tag{8}$$

(4) Geometric compatibility, Fig. 11-13b, requires that the external joint displacements r must equal the element joint displacements $\bar{v}$.

$$\{r_i\} = \{\bar{v}_i\}_{shell} = \{\bar{v}_i\}_{plate} = \{\bar{v}_i\}_{beam} \tag{9}$$

(5) The structure stiffness matrix K for the entire structure can now be assembled by properly adding the element stiffness submatrices of Eq. (6) and (7).

$$\{R\} = [K]\{r\} \tag{10}$$

The K matrix is banded, of the tri-diagonal form and is well conditioned for solution of the unknown joint displacements r.

(6) Internal forces and displacements are then calculated at selected points in each structural element by expressions relating these quantities to the joint displacements.

Assumptions at Shell-Plate-Beam Juncture

Where extensions of the shell and plate elements into the beam all meet at the centroid of the beam cross-section, Fig. 11-14a, they are often assumed to extend from centroid to centroid of beam elements. However, for a general case, Fig. 11-14b, they should be assumed to terminate at the beam edge, Points b and c. For this case the centroid of the beam, Point a, is still assumed to be the longitudinal joint of the system and a suitable transformation of the shell and plate stiffnesses from Points b and c to Point a is necessary. A detailed derivation of the necessary equations for this type of transformation has been presented by Powell[25] and has also been discussed by Jenkins.[6] It is based on the assumption that plane sections remain plane for the beam cross-section under axial, flexural and torsional deformations and that the shear center and centroid of the beam coincide.

Shell Element Stiffness Matrix

The direct stiffness procedure discussed above is independent of the method chosen for determining the element stiffness matrices, so that in a general computer program any subroutines desired can be incorporated for this purpose.

For circular shell elements, a variety of solutions have been proposed based on classical shell theory. Depending on the terms neglected in the derivation one obtains the formulations of Flugge, Dischinger, Aas-Jakobsen, Lundgren, Parme (ASCE Manual 31), Holand, Donnell-Von Karman-Jenkins (DKJ), Finsterwalder, or Schorer. Discussions of these theories is beyond the scope of this paper, but may be found in the books by Billington,[1] Gibson,[2] and Ramaswamy.[4] The most commonly used theories because of their greater simplicity have been the DKJ theory which is accurate for short and intermediate shells, but not for long shells and the Schorer theory which assumes $M_x = Q_x = M_{xy} = 0$ in Fig. 11-6 and thus is accurate for long shells only. Ramaswamy[4] states that the Schorer theory should be used only for shells with span/radius, $L/R > \pi$ and the DKJ theory for $L/R < 1.6$. The author's experience with the DKJ theory indicates a somewhat greater range of accuracy. For single shells, free at the edges and under uniform load, Fig. 11-9b, sufficiently accurate numerical results for design have been obtained for $L/R < 5$, while for a typical interior shell, fixed at the edges and under uniform load, Fig. 11-9c, this reduces to $L/R < 2.5$.

A completely laid out approach for the development of the circular shell element stiffness matrix based on the DKJ theory is given by Gibson.[2] Ramaswamy[4] gives the necessary equations for the development based on either DKJ or Schorer theory.

Plate Element Stiffness Matrix

The response of a plate element, Fig. 11-11b, can be divided into "slab action" due to forces normal to the plane of the plate and "membrane action" due to forces in the plane of the plate. These two actions are independent of each other and thus the 4 x 4 slab stiffness matrix k_s and the 4 x 4 membrane stiffness matrix k_m can be calculated separately to form the 8 x 8 plate element stiffness matrix.

$$\left\{\begin{array}{c} S_s \\ \hline S_m \end{array}\right\} = \left[\begin{array}{c|c} k_s & 0 \\ \hline 0 & k_m \end{array}\right] \left\{\begin{array}{c} v_s \\ \hline v_m \end{array}\right\} \tag{11}$$

The determination of the stiffness matrices can be based either on the "ordinary theory" or the "elasticity theory."[18] The ordinary theory assumes that the membrane stresses in each plate can be calculated by elementary beam theory and that slab bending is defined by means of transverse one-way slab action only, thus in Fig. 11-6, $M_x = Q_x = M_{xy} = 0$ is assumed. The elasticity theory, which is a more accurate approach, utilizes plane stress elasticity theory and classical two-way thin plate bending theory to determine the membrane stresses and slab moments in each plate. Formulas based on both ordinary and elasticity theory for the stiffness matrix coefficients for isotropic, linearly elastic plates having rectangular transverse cross-sections have been summarized for easy usage by De Fries-Skene and Scordelis.[18] The elasticity theory coefficients are taken from the original derivations by Goldberg and Leve.[26]

For the above case there is little reason to use the ordinary theory when a computer program is available for a solution by the elasticity theory, except perhaps for comparative studies. However, direct application of the elasticity theory to cases other than the isotropic, linearly elastic plates of rectangular cross-section becomes exceedingly complex and resort must be made to simpler approaches. For example, for plates having transverse cross-sections that are trapezoidal, element stiffness matrices can be developed using ordinary theory rather readily.

A theory known as the "finite strip method" has great potential for use in problems for which solutions by the elasticity theory are difficult to formulate and solve. The stiffness matrix for an orthotropic plate has been developed by Cheung[23] using this technique. Willam and Scordelis[24] have used the same method to derive a stiffness matrix for an orthotropic plate with closely spaced eccentric ribs in both the longitudinal and transverse directions. The finite strip method may be thought of as a special form of the finite element method. It approximates the behavior of each plate by an assemblage of longitudinal finite strips for which selected displacement patterns varying as harmonics longitudinally and as polynomials in the transverse direction are assumed to represent the behavior of the strip in the total structure. With this assumption the displacement at any point in the strip can be expressed in terms of the eight nodal point displacements shown in Fig. 11-11b. Using successively the strain-displacement relationships, the stress-strain law and thence either the principle of virtual displacements or the principle of minimum total potential energy, the element stiffness matrix and generalized or consistent loads can be derived for the strip.

Beam Element Stiffness Matrix

Elementary beam theory is used to formulate the beam stiffness matrix which relates the four external forces S_i and the four corresponding displacements v_i shown in Fig. 11-11c. It is assumed that plane sections remain plane, warping due to torsion is neglected, and the centroid and shear center of the cross-section coincide. Derivations with formulas for the beam stiffness coefficients have been presented by Powell[25] and Jenkins.[6]

Direct Stiffness Method for Continuous Structures

For continuous shells two approaches can be used which take advantage of the direct stiffness harmonic analysis concept used for simple spans.

In the first approach, originally presented by Morice,[19] the method can be extended to the solution of a prismatic shell span having boundary conditions at the two ends of fixed-fixed, fixed-simple or fixed-free, Fig. 11-15, by replacing the harmonic functions used for simple spans by basic functions which may be written in the form

$$F(x) = A_n(\cosh \alpha_n x - \cos \alpha_n x) - (\sinh \alpha_n x - \sin \alpha_n x) \tag{12}$$

Like harmonic functions, the basic functions are orthogonal, however, they repeat themselves only after four differentiations rather than two as in the case of harmonic functions. The substitution of Eq. (12) into the boundary condition equations permits an evaluation of a sequence of values for the constants A_n and α_n for various cases, Fig. 11-15. The shapes obtained are exactly the same as the natural modes of vibration of a prismatic beam having similar end boundary conditions.

The applied load and displacements may then be resolved into components of a basic function series in a manner analogous to that used in a Fourier analysis for simple span structures such that

$$r(x) = r_1 F_1(x) + r_2 F_2(x) + \cdots + r_n F_n(x) \tag{13}$$

In using basic functions care must be taken that the governing theory used in developing element stiffness matrices and internal force-displacement relationships permit an uncoupling of the solutions for each term of the series. This will be adequately satisfied if the Schorer theory is used for shell elements, the ordinary theory for plate elements and elementary beam theory for beam elements. It will not be satisfied if the DKJ theory is used for shell elements or the elasticity theory is used for plate elements. Thus it is seen that the basic function approach is limited to long shells.

A second approach, originally developed by Pultar[27] and also programmed by others,[22,28] may be used for prismatic shells continuous over rigid or flexible interior supports, but simply supported at the two extreme ends, Fig. 11-16. A force method of analysis is used in which the redundants are taken as joint and plate interaction forces between the folded plate structure and the supporting interior transverse diaphragm. These may be taken as infinitely rigid or as flexible rigid frames. The joint redundants consist of up to three forces at each longitudinal joint which act in vertical, horizontal and rotational directions, Fig. 11-17. The plate redundants are transverse distributed interaction forces, for example, a set of up to four plate forces for each plate can be used, consisting of distributed normal and tangential forces having triangular variations between the longitudinal edges of the plates, Fig. 11-17. All of the redundant interaction forces are assumed to be uniformly distributed in the span direction over a length equal to the specified diaphragm thickness. The redundant interaction forces are determined as those required to establish compatibility between the folded plate structure and the interior supporting structure at each longitudinal joint in the vertical, horizontal, and rotational directions and in each plate at the third points between joints in directions normal and tangential to the plane of the plate. It is obvious, Fig. 11-17, that a large number of redundants can be involved and that a large number of terms of the appropriate Fourier Series is needed to represent their action on the folded plate structure. Nevertheless, complete solutions of examples similar to that shown in Fig. 11-16 have been solved in less than one minute on a CDC 6400 computer.

Advantages and Disadvantages of Direct Stiffness Harmonic Analysis

Advantages of this method are:

1. It is well suited for computer programming and can yield a complete and accurate solution in a reasonable amount of computer time.
2. Any desired theory can be used to determine the response of the structural elements and a variety of structural elements can be incorporated into the solution easily.
3. Both surface and joint loadings of arbitrary longitudinal variation can be treated.
4. Any combination of displacement and force boundary conditions at the longitudinal joints can be used.

Disadvantages of this method are:

1. It is restricted to structures with simple supports at the extreme ends when a harmonic analysis is used. For long span structures, the use of basic functions permits a solution of certain additional end support conditions.
2. The material and dimensional properties of each structural element making up the cross-section must be constant in the longitudinal direction.

FINITE ELEMENT METHOD

General Remarks

This method has been described extensively in the literature during the past decade. A comprehensive discussion of the theory and application of the method is given in the book by Zienkiewicz.[29] Another paper by Clough and Johnson[30] in these Proceedings discusses its application to arbitrary shells, so that only a brief discussion of its application to prismatic shells will be given here.

In the finite element method the actual continuum is idealized by an assembly of finite elements interconnected at nodal points, Fig. 11-18. For a prismatic shell system, the finite elements may consist of flat or curved two dimensional shell or plate elements and transverse or longitudinal one dimensional beam type elements. Stiffness matrices, which approximate the behavior in the continuum, are developed for the finite elements based on assumed displacement or stress patterns, after which an analysis based on the direct stiffness method may be performed to determine the nodal point displacements and thence the internal stresses in the finite elements. It should be recognized that the accuracy obtained is dependent on the assumptions used in deriving the stiffness matrices and on the fineness of mesh used in subdividing the structure. As generally applied, the results obtained closely satisfy compatibility, but not necessarily equilibrium in the continuum until a sufficiently fine mesh is used.

Shell and Plate Finite Elements

A number of investigators[31-37] have developed two dimensional finite elements specifically for analyzing prismatic shells. A discussion and comparison of some of these have been presented by Kohnke and Schnobrich[37] and by Clough and Johnson.[30] Basic differences in the various investigations are: (1) curved versus flat elements; (2) number of nodal points and number of degrees of freedom per nodal point; and (3) assumed displacement patterns used in developing the element stiffness. On all of these points, opinions differ on what is the best approach.

For folded plates it is obvious that flat elements should be used, however, for curved cylindrical shells the use of flat rather than curved elements involves a geometric error in modeling the structure. Nevertheless, if sufficient flat elements are used it can be shown that this error becomes quite small.

Ideally the elements should have nodal points only at the four corners, Fig. 11-18. This minimizes the band width in the assembled structure stiffness matrix and simplifies the interconnection of beam elements into the system. The number of degrees of freedom used per corner node has ranged from five[33] to twelve.[37] As the number used increases, fewer elements are needed in idealizing the continuum to achieve a given degree of accuracy for a given solution. However, the solution time may be about the same as that using a finer mesh of simpler elements with fewer degrees of freedom at each node, but the same total number of degrees of freedom for the structure. The latter elements have the advantage of greater versatility in varying material and dimensional properties, cut outs, boundary conditions, loads, and the introduction of beam elements and are thus favored by the author.

In developing the element stiffness, displacement patterns should be chosen with the following desirable criteria in mind: (1) constant strain patterns should be included; (2) rigid body displacements should not induce element strains; and (3) compatibility along element interfaces between nodal points should be maintained. Most elements presently being used do not satisfy all of these criteria completely.

Beam Finite Elements

To be of practical use in the design of prismatic shell systems, it is desirable that beam type elements can be incorporated into the finite element analysis. This then makes it possible to study ribbed shells, the effect of flexibility of rigid frame or arch supports as well as many other topics. Meyer and Scordelis[38] have incorporated straight beam elements with flat rectangular plate elements in which both element types have six degrees of freedom per node. Constant strain and rigid body modes are included, but compatibility is violated. Kohnke and Schnobrich[37] have used straight and curved beam elements with curved cylindrical shell elements in which twelve degrees of freedom per node are used. Their formulation achieves compatibility, but violates the rigid body mode criteria.

Advantages and Disadvantages of Finite Element Method

Advantages of this method are:

1. It is the most general method available and can treat arbitrary loadings, boundary conditions, varying materials and dimensional properties in both longitudinal and transverse directions, and cut outs.
2. Beam type transverse and longitudinal ribs as well as rigid frame or arch type supporting structures can be incorporated as integral parts of the structural system.

Disadvantages of this method are:

1. It requires a greater amount of computer time than a direct stiffness harmonic analysis to obtain a solution of comparable accuracy.
2. A refined mesh size must be used to achieve accurate results in the vicinity of steep stress gradients, for example at concentrated loads and reactions.
3. Static equilibrium is not automatically satisfied, but is approached as the mesh size used in the analysis is refined. Judgment must be used in selecting an appropriate mesh layout and in interpreting the results obtained.