TORSION OF CORE STRUCTURES

Fig. 6-2(a) shows a view of a typical core structure which consists essentially of two channel sections coupled at each floor level by lintel beams and floor slabs. It may be assumed that the cross-sectional shape of the structure is maintained by the high in-plane stiffness of the floor slabs surrounding the box.

The torque-rotation characteristics of such structures may be derived using an analysis which is analogous to the continuous medium technique employed earlier for plane walls with openings. It is again assumed that the system of connections is constant throughout the height of the building.

Under the action of an applied torque, the cross-section will undergo a rigid body rotation as indicated in Fig. 6-2(b), with a point of contraflexure occurring at the mid-span position of the connecting beams due to the anti-symmetry of the mode of deformation.

If the structure is assumed 'cut' along the line of contraflexure, the only forces acting there will be shear forces, which may be replaced by a continuous shear distribution of the discrete set of connecting beams is again replaced by an equivalent continuous medium. These shear forces Q are equal in magnitude, but act in opposite senses on opposite connecting beams, as shown in Fig. 6-2(d). In order to achieve compatibility at the cut position in the complete structure, these shear forces must be of such a magnitude that they produce vertical displacements which are equal and opposite to those produced by the torsional rotation. (Fig. 6-2(c)) Following a procedure similar to that of Michael,[2] the compatibility equations may be combined with the torque-rotation and the bending moment-curvature relationship to produce a single third order governing equation.

Using the notation of Fig. 6-2(b) and (c), the displacements in the O_{yx} and O_{zx} planes are, respectively,

$$v_2 = v_1 = \frac{1}{2} D\theta \, ; \quad w_2 = w_1 + \frac{1}{2} B \frac{dv}{dx} = \frac{1}{4} BD \frac{d\theta}{dx}$$
$$u_2 = u_1 = \frac{1}{2} B\theta \, ; \quad w_3 = w_2 + \frac{1}{2} D \frac{du}{dx} = \frac{1}{2} BD \frac{d\theta}{dx} \tag{9}$$

The displacement v of the shear center S of each channel is,

$$v_s = \left(\frac{1}{2} D + r \right) \theta \tag{10}$$

The torque-rotation relationship for the system of channels is,

$$T = 2V \frac{d\theta}{dx} - 2W \frac{d^3\theta}{dx^3} + F\,(D + 2r) \tag{11}$$

where T is the applied twisting moment, F is the horizontal shear force, at the shear center, in each channel, and V and W are the St. Venant torsion and End Warping constants respectively.[3] For a channel section, V and W become,

$$V = G \frac{t^3}{3} (B + D - b - t)$$

$$W = \frac{EtB^2}{192} (D - b - t)^3 \left(1 + \frac{tB^3}{4I_w}\right)$$

where t is the wall thickness, and b is the clear span of the opening.

The shear force F produces a bending moment per unit height m'_x, equal in magnitude to F. Thus,

$$m'_x = \left(T - 2V \frac{d\theta}{dx} - 2W \frac{d^3\theta}{dx^3}\right) / D + 2r \tag{12}$$

The shear forces Q are of such a magnitude that they produce a deflection at the point of contraflexure which is equal and opposite to the deflection w_3 produced by the rotation, so that,

$$w_3 = \frac{Qb^3}{24EI_b} \tag{13}$$

The upwards and downwards shear forces produce a restoring bending couple on the channel in the O_{xy} plane, of magnitude QB (Fig. 6-2(d)). The intensity of the bending moment in the continuous system is thus,

$$m''_x = \frac{24EI_b \, BW_3}{hb^3} \tag{14}$$

The net bending moment intensity m_x which must be resisted by the wall is the difference between the externally applied couple m'_x and the restoring couples m''_x.

$$m_x = m'_x - m''_x \tag{15}$$

Referred to the shear center, the moment-curvature relationship for the wall is,

$$m_x = -EI_w \frac{d^2v}{dx^2} = -EI_w \left(\frac{1}{2} D + r\right) \frac{d^2\theta}{dx^2}$$

so that the moment intensity per unit height is,

$$m_x = \frac{dm_x}{dx} = -EI_w\left(\frac{1}{2}D + r\right)\frac{d^3\theta}{dx^3} \tag{16}$$

Hence, from Eq. (15), (16), (14), (12) and (9), the governing equation becomes,

$$\frac{d^3\theta}{dx^3} - \gamma^2\frac{d\theta}{dx} = \beta T \tag{17}$$

where

$$\gamma^2 = \frac{4V + \dfrac{24\,EI_b B^2 D\,(D + 2r)}{hb^3}}{4W + EI_w\,(D + 2r)^2}$$

and

$$\beta = \frac{2}{4W + EI_w\,(D + 2r)^2}$$

In the case of a concentrated twisting moment of magnitude T_i applied at any level x_i,

$$T = T_i < x_i - x >^o \tag{18}$$

If the core is rigidly built-in at the base, then

$$\text{at} \quad x = 0, \quad \theta = 0 \quad \text{and} \quad \frac{d\theta}{dx} = 0 \tag{19}$$

Since the bending moment intensity in the continuous medium is zero at the top of the walls,

$$\text{at} \quad x = H, \quad \frac{d^2v}{dx^2} = 0 \quad \text{and} \quad \frac{d^2\theta}{dx^2} = 0 \tag{20}$$

The solution of Eq. (17), subject to the boundary conditions (19) and (20), may be shown to be,

$$\theta = -\frac{\beta T_i}{\gamma^2}\left\{<x_i - x>^1 - \frac{1}{\gamma}<x_i - x>^0 \sinh\gamma(x_i - x) - x_i\right.$$

$$+ \frac{1}{\gamma\cosh\gamma H}[\sinh\gamma H + \sinh\gamma(H - x)(\cosh\gamma x_i - 1)$$

$$\left. - \sinh\gamma(H - x_i)]\right\} \tag{21}$$

and the bending moment on the walls is given by,

$$M_x = -EI_w\left(\frac{D}{2} + r\right)\frac{d^2\theta}{dx^2}$$

$$= \frac{1}{2}EI_w(D + 2r)\frac{\beta T_i}{\gamma}\left\{<x_i - x>^0 \sinh\gamma(x_i - x)\right.$$

$$\left. + \frac{\sinh\gamma(H - x)}{\cosh\gamma H}(\cosh\gamma x_i - 1)\right\} \tag{22}$$

The other forces follow from the earlier relationships.

Eq. (21) describes completely the relationship between a unit twisting moment at any level x_i and the rotation produced at level x, enabling a complete set of influence coefficients k_{ij} (i.e. rotation at x_i due to unit twisting moment at x_j) to be evaluated readily for any prescribed set of reference levels.

The above analysis tacitly assumes that the core structure is symmetrical. If this is not the case, the displacement relationships Eq. (9) and (10) must be modified to take account of the asymmetry, although the general technique remains unaltered.

TORSION OF PLANE WALLS

The simple engineering theories of bending and torsion indicate that, for a thin rectangular cross-section, of depth d and thickness t, the bending stiffness (I) is proportional to d^3t and the torsional stiffness (J) is proportional to dt^3. In practical shear wall structures, the depth d is very much greater than the wall thickness t, with the result that the torsional stiffness is very much less than the bending stiffness. The effect of the former may thus often be neglected, with little error involved.

If warping effects are neglected, and ordinary simple torsion theory is used, the torque-rotation relationship for a wall element is,

$$T_i = C_i\left(\frac{d\theta}{dx}\right)_i \tag{23}$$

where C_i is the torsional stiffness of the element at that level. For a plane wall which has the form of a thin rectangle,[3]

$$C = GJ = G.\ \frac{1}{3}\, dt^3$$

where G is the shear modulus.

If the wall has 'flanges,' the torsional stiffness of the complete section is obtained by summing the stiffnesses of the component rectangular elements.[3]

If the wall is perforated by a regular series of openings, the reduction in torsional stiffness is given approximately by the curve of Fig. 6-3. This graph is based on a series of torsional tests on thin perspex specimens, 12 in. long between end fixings, 1/2 in. thick, and from 2 to 6 in. wide, containing a regular set of 12 openings. After testing the unperforated specimen to give a datum result, the hole sizes were gradually increased and the reduction in stiffness observed. The graph is not complete, since it does not include the influence of coupling beam stiffness, although the tests indicated that, over a typical range of relative stiffnesses, they do not greatly alter the results. In addition, the restraining effect of floor slabs was not simulated. However, the curve is included in view of the complete lack of information on this topic, and should be useful in giving a guide to the likely reduction in torsional stiffness caused by typical patterns of openings.

If the torsional stiffness is constant, Eq. (23) may be integrated directly, and, on putting in the boundary condition of zero twist at the base, the rotation θ at any level x due to an applied torque T_i at level x_i becomes,

$$\theta = \frac{T_i}{C} \left\{ x - \langle x - x_i \rangle^1 \right\} \tag{24}$$

A Macaulay bracket is again used to enable the single expression Eq. (24) to describe the behavior above and below the point of load application. The angle of rotation increases linearly up to level x_i, and remains constant thereafter.

Alternatively, by using the finite difference equivalent for the rate of change of twist, the twisting moment at x_i becomes.

$$T_i = C_i \frac{1}{2s} (\theta_{i+1} - \theta_{i-1}) \tag{25}$$

where 's' is the height interval between reference levels. Eq. (25) may be used for all levels except the top-most level x_n, where the alternative backward difference expression must be used,

$$T_n = C_n \frac{1}{s} (\theta_n - \theta_{n-1}) \tag{26}$$

ANALYSIS OF COMPLETE STRUCTURE

Suppose that the structure consists of a number of parallel coupled wall assemblies and individual cantilever elements. In either case, for the k^{th} wall unit, the load-deflection relationship may be expressed in matrix form as

$$Y_k = F_k P_k \tag{27}$$

where Y_k and P_k are column vectors of deflections y_{ik} and total applied loads P_{ik} at the set of chosen reference levels x_i, and F_k is a square flexibility matrix of influence coefficients f_{ij_k} evaluated from Eq. (7) or (8). Relationships of the form of Eq. (27) may be set up for each wall assembly. In buildings of the form considered, there is usually a certain amount of repetition in the layout, so that the number of different types of wall assemblies is limited. Any applied distributed loads on the structure may be considered as a series of point loads at the reference levels.

For a perforated core element, the torque-rotation relationship may be expressed in the form,

$$\theta = K_k T_k \tag{28}$$

where θ and T_k are column vectors of rotations θ_i and total twisting moments T_i at the same set of reference levels, and K_k is a square matrix of influence coefficients k_{ij_k} evaluated from Eq. (21).

For plane walls, a similar relationship can be set up using Eq. (24).

Alternatively, it could be set up in the inverse form,

$$T_k = C_k \theta \tag{29}$$

where T_k and θ are again column vectors of overall twisting moments and rotations, and C_k is a square matrix of coefficients derived from Eq. (25) and (26). In that case, the same expressions which follow may be derived, provided that the matrix K_k^{-1} is replaced by C_k.

It is assumed that the floor slabs are so stiff in their own plane that they undergo only rigid body displacements, so that the structure will deform in plan view as shown in Fig. 6-4. For convenience, all displacements are referred to the left hand end, 0, and the displacement of any element at level x_i at a distance z_k from the datum or 'center of rotation' will thus consist of a deflection $(y_i + \theta_i z_k)$ and a rotation θ_i. Eq. (27) then becomes,

$$Y_k = \bar{y} + \bar{\theta} z_k = F_k P_k \tag{30}$$

where $\bar{y}$ and $\bar{\theta}$ are column vectors of the deflection of the datum position and the rotation at each reference level.

Suppose that the structure is subjected at each level to a load P_i, applied at the datum, and a twisting moment T_i. (Any set of applied forces and moments on a rigid body may always be resolved into a force and couple at any specified position on the body.) These forces must be resisted by the wall assemblies, so that, for horizontal and rotational equilibrium,

$$\bar{P}_i = \Sigma P_{ik} \tag{31}$$

$$T_i = \Sigma P_{ik}\, z_k + \Sigma T_{ik} \tag{32}$$

where P_{i_k} and T_{i_k} are the horizontal load and twisting moment carried by wall assembly k at level x_i, and the summations are carried out over all wall assemblies. For the complete structure, the equilibrium equations become,

$$\bar{P} = \Sigma \bar{P}_k \tag{33}$$

$$\bar{T} = \Sigma \bar{P}_k\, z_k + \Sigma \bar{T}_k \tag{34}$$

where $\bar{P}$ and $\bar{T}$ are the column vectors of the total applied load and twisting moment at each level. Substitution of Eq. (30), (28) and (29) into (33) and (34) yields, respectively,

$$\bar{P} = \Sigma \bar{F}_k^{-1} (\bar{y} + \bar{\theta}\, z_k) \tag{35}$$

$$\bar{T} = \Sigma \bar{F}_k^{-1} (\bar{y} + \bar{\theta}\, z_k)\, z_k + \Sigma \bar{K}_k^{-1} \bar{\theta} \tag{36}$$

In Eq. (36), it is tacitly assumed that the appropriate forms of the flexibility matrices $\bar{K}_k$ from either (21) or (24) are used in the analysis.

The solution of Eq. (35) and (36) is,

$$\begin{aligned} \bar{y} &= [\bar{G}_1 - \bar{G}_2\, \bar{G}_3^{-1}\, \bar{G}_2]\ [\bar{P} - \bar{G}_2\, \bar{G}_3^{-1}\, \bar{T}] \\ \bar{\theta} &= [\bar{G}_3 - \bar{G}_2\, \bar{G}_1^{-1}\, \bar{G}_2]\ [\bar{T} - \bar{G}_2\, \bar{G}_1^{-1}\, \bar{P}] \end{aligned} \tag{37}$$

where
$$\bar{G}_1 = \Sigma \bar{F}_k^{-1}$$
$$\bar{G}_2 = \Sigma \bar{F}_k^{-1}\, z_k$$
$$\bar{G}_3 = \Sigma (\bar{F}_k^{-1}\, z_k^2 + \bar{K}_k^{-1})$$

For generality, the solution has been given for both an applied load and an applied twisting moment at each reference level. If torsion alone is considered, the matrix $\bar{P}$ is set to zero.

Once the deflections and rotations have been determined, the loads and twisting moments on the various wall elements at the different levels may be evaluated from Eq. (27) and (28). The forces and stresses follow from the earlier equations.

MODEL TESTS

Three twenty-story models were tested. The models were fabricated from sheet perspex, with plan forms shown in Fig. 6-5 and 6-6. The models were constructed by cutting slots in the floor slabs to accommodate the walls which were made in one piece and glued into position, using blocks to maintain a constant story height. Model 3 was constructed by glueing flank walls (D) and cross-wall 'flanges' (E) to Model 2. The nominal dimensions and properties of the models are given in Table 6-1.

All models were asymmetrical, so that both bending and torsional deformations would generally result from any applied force system. The first model was a simple idealised structure designed to be torsionally weak, whilst the second and third were designed to be more realistic structures resembling the basic form of cross-wall assembly encounted in an apartment block.

The walls and cores were glued into a 25.4 mm (1 in.) perspex sheet at foundation level. The models were then clamped horizontally in a test frame, with steel sections employed to brace the foundation slab to give a condition representing as closely as possible a rigid foundation. Throughout the tests, checks were carried out to ascertain if any foundation movement was taking place.

Torsional moments were applied by a superposition of two equal and opposite loads, applied near the edges by dead weights through wires attached to the floor slabs. The loads were applied in increments, and strains and displacements measured by electrical resistance gages and dial gages respectively; they were then plotted so that unit values could be obtained from the optimum linear curve.

COMPARISON BETWEEN THEORY AND EXPERIMENT

A comparison is made between theoretical and experimental results in Fig. 6-7 to 6-12. All curves refer either to a uniform eccentric line load, of 1 kg per story height, or a uniformly distributed torsional moment, produced by two equal and opposite line loads of 1 kg per story applied at the positions indicated in Fig. 6-5 and 6-6.

Fig. 6-7 shows the rotational deformations due to a uniform twisting moment, and Fig. 6-8 shows typical deflection profiles, for Model 3 only, due to a uniform line load. Fig. 6-9 and 6-10 show the bending stress distributions due to torsion, while, for completeness, Fig. 6-11 gives the corresponding stress distributions due to an eccentric line load. In each case, the bending strains were measured just above the third floor level, this being chosen to ensure that the stresses would be affected as little as possible by any localised foundation effects. Similar results were obtained from Models 1 and 3 subjected to line loads, and are not reproduced here.

The shear strains were measured at the fourth story level by 45 deg strain gage rosettes. Being relatively small, these were more difficult to measure accurately, and the results are shown only for Model 2 subjected to torsional loading, since these appeared to be the most consistent and accurate of the three tests.

Although the theory predicts accurately the maximum deflections due to eccentric line loads, it underestimates the torsional stiffness of the structure, the theoretical rotations being greater than those measured. The same effect is shown in the stress distributions. Consequently, the results are less accurate for structures subjected to torsional moments, although the agreement between theory and experiment is still reasonable. The shear stress distributions indicate that the theory is capable of predicting accurately the distribution of load between the various elements of the structure.

Earlier work (e.g. Reference 1) has shown that the continuous connection method is capable of yielding accurate results, for both deflections and stresses, for laterally loaded plane perforated shear walls. Consequently, the discrepancy between theory and experiment is probably due to errors involved in combining such plane elements into a three-dimensional assembly. It is not possible to produce easily a small-scale three-dimensional model of the mathematical conception consisting of plane coupled elements constrained to act together with rigid-body movements, in plan, as envisaged in Fig. 6-4; it is necessary in the construction of the model to use floor slabs which are rigidly joined to the vertical wall elements, and they must undergo out-of-plane deformations when the walls bend under the action of lateral loads. Consequently, in assessing the coupling action between wall elements, it is necessary to estimate the effective bending stiffness of the continuous floor slab. For simplicity in construction of the present models, no lintel beams were provided, and the only connecting medium between vertical elements was the horizontal slabs.

In the theoretical work, the effective width of floor slab connecting plane walls was determined from the curves given by Qadeer and Stafford Smith.[4] These are limited in use, and the effective width of floor slab coupling the box core C and plane wall B in Models 2 and 3 (Fig. 6-6) could only be estimated, and some error may be present. In addition, the forms of the stress distributions in the two coupled wall elements in the simplest model tested (Fig. 6-9) show that more of the applied moment is carried by axial forces in the walls than is predicted theoretically, indicating an underestimation of the coupling stiffness.

When rotation of the structure takes place, the floor slabs, being constrained to act with the wall elements, will tend to warp as well as bend out of their plane. In its present form, the theory does not take account of the warping stiffness of the floor slabs, which in the models tested may afford a considerable restraint and increase the torsional stiffness of the system. It is likely that the discrepancies between theoretical and experimental results are due largely to this effect.

In practice, however, the floor slabs may be constructed from precast units, with in-situ joints at the walls. Alternatively, if the floor slabs are cast in-situ, the joints are often designed to provide a shear connection but to provide little or no bending connection. This is done so that localised spalling of the concrete due to high connecting moments may be avoided. In either case, the moment connections are weak and the effective coupling stiffness of the floor slabs is low. The main connections between shear wall elements will then be the lintel beams spanning between them at each floor level, whose stiffness can be determined accurately. The influence of bending and warping of the floor slabs will then be relatively much smaller, and the theory presented is likely to give much more accurate predictions of the deflections and stresses throughout the structure.

The continuous connection technique on which the analysis is based becomes more accurate as the number of stories increases. It has been shown that in the analysis of multi-story structures it is essential to include the effects of axial deformations of the vertical elements, and this is included in the analysis. Since the analysis is applied to a continuous system, there is no need to refer to every level as in the case of a conventional frame analysis. The engineer can thus decide on the number of reference levels which need be employed, and a standard program could be adopted to perform the analysis for a fixed number of divisions of the total height, for example 10. The orders of the matrices required are very small, since the order of the largest matrix handled is the same as the number of reference levels used. In fact, for preliminary design calculations, approximate analyses of complex structures may be performed by hand calculation if only a small number of reference levels are employed.

CONCLUSIONS

A simple approximate method has been presented for the torsional analysis of three-dimensional structures consisting of assemblies of coupled shear walls and core elements. Even if the system of walls is complex, it is generally possible to divide them into assemblies of discrete coupled units for analytical purposes. The method is aimed particularly at industrialised building systems, including large panel construction, in which there tends to be an essential regularity of structural form throughout the height. However, a limited number of changes in geometrical or stiffness characteristics, along with different foundation conditions, may readily be incorporated.

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