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Similitude Requirements for Dynamic Models

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Synopsis: This paper is a state-of-the-art discussion on dynamic modeling theory and its application to scale modeling of reinforced concrete structures. Dimensional analysis is used to develop similitude requirements according to which the geometry, initial and boundary conditions, material properties and the loading of the model and prototype have to be related so that the behavior of the latter can be expressed as a function of the behavior of the former. The discussion focusses on various types of models that can be utilized to simulate the dynamic response of structures in the elastic and inelastic range. Commonly encountered problems in achieving satisfactory similitude are outlined and assessed as to the effect they may have on the reliability of prototype response prediction.

Keywords: dynamic response; dynamic structural analysis; earthquake resistant structures; elastic analysis; failure; failure mechanisms; loads (forces); models; reinforced concrete; scale (ratio); structural analysis.

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INTRODUCTION

Experimentation with reduced-scale physical models is often a feasible means for predicting the response of structural components, subassemblies and complex structures subjected to gravitational effects, wind or seismic effects, impacts, pressurization, settlements, time and temperature effects, or any combination thereof. The reliability of response prediction depends strongly on the ability to reproduce, at model scales, all prototype features that affect the quantities or parameters which are to be predicted through the model test. These prototype features may vary with the purpose of the model test and consequently may lead to different modeling requirements for static vs. dynamic models, linear elastic vs. ultimate strength models, or models intended for the study of temperature dependent effects.

This paper reviews briefly the principles of modeling theory applicable to the general problem of physical modeling. Emphasis is then placed on developing similitude requirements for various types of ultimate strength models of reinforced concrete structures under the influence of gravitational and dynamic effects. Such models are of much interest in earthquake engineering where post-elastic response characteristics and safety against failure are of primary importance, and where laboratory experimentation of complete structures is in most cases limited to testing at reduced scales, often at rather small scales. While in most cases it is relatively easy to develop theoretical model similitude requirements, great difficulties are usually encountered in fulfilling all of these requirements and certain distortions must often be accepted. Nevertheless, it is often possible to minimize the influence of these distortions or to evaluate their effects on the predicted response, thus, permitting a quantitative evaluation of model test results in the prototype domain. Several of the problems encountered in fulfilling similitude requirements are discussed in this paper.

MODELING THEORY

Modeling theory establishes the rules according to which the geometry, material properties, initial conditions, boundary conditions and external influences (loading) of the model and prototype have to be related so that the behavior of the latter can be expressed as a function of the behavior of the former. The purpose of modeling theory is to develop a complete independent set of correlation functions (similitude laws) defining the model-to-prototype correspondence. Well established principles of dimensional analysis [1,2] can be utilized to derive these correlation functions.

Dimensional analysis is based on the premise that every physical phenomenon can be expressed by a dimensionally homogeneous equation of the type

$$q_1 = F(q_2, q_3, \ldots q_n) \tag{1}$$

where n is the total number of physical quantities involved in the phenomenon, q_1 is the quantity which is to be evaluated and q_2 to q_n are the quantities on which q_1 depends. According to Buckingham's Pi theorem, every dimensionally homogeneous equation involving n physical quantities can be reduced to a functional relationship between a complete set of n-N independent dimensionless products (Π-factors) taking the form

$$\Pi_1 = f(\Pi_2, \Pi_3, \ldots \Pi_{n-N}) \tag{2}$$

where Π_1 to Π_{n-N} are independent dimensionless products of powers of the physical quantities q_1 to q_n. The number N is the rank of the dimensional matrix and is usually equal to the number of basic units needed to describe the physical quantities.

Since Eq. (2) is identical to Eq. (1), it describes the same physical phenomenon and, because of its dimensionless form, must be equally valid for prototype and model if similitude is to be achieved. A sufficient condition for complete similitude is therefore

$$\begin{aligned}(\Pi_1)_p &= (\Pi_1)_m \\ (\Pi_2)_p &= (\Pi_2)_m \\ &\vdots \\ (\Pi_{n-N})_p &= (\Pi_{n-N})_m\end{aligned} \tag{3}$$

where the subscripts p and m refer to prototype and model, respectively. The first of these equations is often referred to as the prediction equation while the other equations constitute the design conditions for the model.

To assure a reliable prediction of the response quantity q_1 and, at the same time, to minimize the effort in constructing a physical model, extreme care must be exercised in specifying the right number of physical quantities in Eq. (1). Quantities which have insignificant effect on q_1 will impose unnecessary constraints on the model design, while neglecting a significant quantity may yield incorrect results. For instance, when only mode shapes and frequencies in the linear elastic range are of interest, a simulation of gravitational acceleration and stress histories is not needed while proper simulation of boundary conditions is essential.

If all similitude requirements (Eq. 3) are fulfilled, the model is often referred to as a true model. In distorted models, where one or more of the similitude requirements are violated, the effects of the distortions must be accounted for in the response prediction. Corrections for a violation of one similitude law can often be made through adjustments of other similitude laws. If only one prediction quantity is of primary interest (e.g., failure load), it is often possible to evaluate the distortion of the predicted value through extrapolation from several model tests at different scales, provided that the failure mode does not change.

The derivation of a complete and independent set of Π-factors is facilitated by the use of the dimensional matrix which is a matrix containing dimensional exponents of the physical quantities expressed in terms of independent basic quantities. The choice of basic quantities is not unique but it is customary to use either force, length, time and temperature (FLT system) or mass, length, time and temperature (MLT system) for this purpose. When the dimensions of the relevant physical quantities are properly arranged in a dimensional matrix, it is reasonably simple to extract dimensionless products by comparing individual quantities as to their dimensional dependence. To assure that these products constitute a complete and independent set and that they permit optimum control of the design and execution of the model experiment, it is helpful to follow a series of guidelines which are discussed in detail in Ref. 3.

An important conclusion can be drawn from the observation that all physical quantities can be expressed in terms of independent basic quantities. Consequently, as many quantities can be scaled arbitrarily (within physical constraints) as there are basic quantities needed to describe the problem. All other quantities are then scaled as functions of these selected scales. The arbitrarily scaled quantities must be independent, but other than that, can be chosen by the model designer.

In order to illustrate the derivation of similitude requirements, let us consider a problem in which it is desired to predict the stress response of a structure under dynamic actions such as an earthquake ground motion. In the simplest case, the stress history σ will be a function of a position vector $\vec{r}$, the time t, the mass density ρ, the material stiffness E, the acceleration time history a, the gravitational acceleration g, a characteristic length parameter ℓ, and initial conditions which may be described by σ_o and $\vec{r}_o$. Thus, Eq. (1) can be written for this case as

$$\sigma = F(\vec{r}, t, \rho, E, a, g, \ell, \sigma_o, \vec{r}_o) \qquad (4)$$

When needed, this relationship can be expanded to include other physical quantities that may enter the problem under consideration.

The dimensional matrix in the MLT system (neglecting dependence on temperature) is of the following form:

	σ	$\vec{r}$	t	ρ	E	a	g	ℓ	σ_o	$\vec{r}_o$
M	1	0	0	1	1	0	0	0	1	0
L	-1	1	0	-3	-1	1	1	1	-1	1
T	-2	0	1	0	-2	-2	-2	0	-2	0

Products of powers of the ten physical quantities can now be assembled to arrive at n-N = 10-3 = 7 independent dimensionless terms (Π-factors). In general, the choice of the dimensionless terms is not unique which may affect the similitude requirement and consequently the model design. A suitable set of Π-factors for this example, arranged in the form of Eq. (2), is

$$\frac{\sigma}{E} = f\left(\frac{\vec{r}}{\ell}, \frac{t}{\ell}\sqrt{\frac{E}{\rho}}; \frac{a}{g}, \frac{a\ell\rho}{E}, \frac{\sigma_o}{E}, \frac{\vec{r}_o}{\ell}\right) \tag{5}$$

To achieve true simulation of the stress history by means of a physical model, each term in this equation must be equal in model and prototype.

Equation (5) can be taken as a basis for deriving similitude requirements for a large number of modeling problems involving dynamic effects. However, for some problems (such as linear elastic problems) too many physical quantities are included in this equation which may permit a relaxation of certain similitude requirements. For other problems it may be necessary to include other quantities (such as temperature, viscosity, coefficient of friction, etc.) that affect the stress history, which would lead to additional similitude requirements. A closer examination of Eq. (5) will also show that for ultimate strength models in reinforced concrete it will hardly ever be possible to fulfill all of the similitude requirements for a true model unless gravitational effects are negligible. In addition, by omitting all material properties other than the stiffness E it is implicitly assumed that time and size dependent effects are properly simulated in the component materials as well as at the interfaces between the materials (bond). Thus, a multitude of problems exist in designing feasible models and in simulating material properties. Nevertheless, in many practical model studies it is possible to overcome these difficulties and obtain a reliable prediction of the prototype response. It is the objective of the following discussion to suggest various types of models that may be suitable for predicting dynamic response characteristics, and to outline problems in achieving similitude and suggest solution techniques where possible.

DYNAMIC MODELS OF REINFORCED CONCRETE STRUCTURES

True Ultimate Strength Models

If all physical quantities listed in Eq. (4) have a significant effect on the stress history, the similitude requirements for a true model can be derived directly from Eq. (5). Since three independent basic quantities (M,L,T) are needed to describe the problem, three physical quantities can be selected for independent scaling. Practically, only two quantities can be selected freely since the gravitational acceleration g is equal in model and prototype, i.e., $g_m/g_p = 1$. If ℓ and E are selected for independent scaling in addition to g, the similitude requirements shown in column (1) of Table 1 need to be fulfilled for a true model. The subscript r used in this table refers to the ratio of units defining the model to prototype relationship for a physical quantity (e.g., $\ell_r = \ell_m/\ell_p$).

The most restrictive requirement comes from the term $a\ell\rho/E$ in Eq. (5) which necessitates that $(E/\rho)_r = \ell_r$, since $a_r = g_r = 1$. This equation requires that model materials have a much smaller stiffness or much larger density than the prototype material. In reinforced concrete, where prototype and model materials are usually of similar stiffness and density, this all but eliminates the use of true models for dynamic studies involving post-elastic behavior and gravitational effects.

In theory it is possible to test true ultimate strength models, even when prototype-like materials are used, by performing model tests in centrifuges capable of generating high uniform acceleration fields. This will permit full control over the scaling of the gravitational acceleration g and elminate the requirement $(E/\rho)_r = \ell_r$. If prototype material is used, $(E/\rho)_r$ is equal to one and g_r has to be equal to ℓ_r^{-1} which would require very large accelerations for small-scale models.

Ultimate Strength Models With Artificial Mass Simulation

By far the most practical approach to dynamic modeling of reinforced concrete structures is to augment the density of the structurally effective model material with high density material which is structurally not effective but permits the fulfillment of the similitude requirement $(E/\rho)_r = \ell_r$. Practically, this will require the addition of suitably distributed weights which must be attached to the structural elements in a manner that does not change the strength and stiffness characteristics. Since addition of weights will alter the local stress histories close to attachment points, such models are not true models but will provide fully satisfactory results provided that the weights are properly distributed and are attached to points of noncritical stress histories.

If a correct simulation of the mass distribution in space is essential, it will be necessary to decouple the mass density ρ_o of the structurally effective material from an additive ρ_1 which is to be built into the model but has no component in the prototype. From the term $a\ell\rho/E$ in Eq. (5), the mass density ρ_1 can be derived as

$$\rho_1 = \left[\frac{E_r}{\ell_r} - (\rho_o)_r\right](\rho_o)_p \qquad (6)$$

For instance, for a 1:4 scale model using prototype material $[E_r = (\rho_o)_r = 1]$ the density will have to be increased by a factor of three.

For line elements the amount of mass μ_1 per unit length that must be added to the model is given by

$$\mu_1 = [E_r \ell_r - (\mu_o)_r](\mu_o)_p \qquad (7)$$

where μ_o is the mass per unit length of the structurally effective material.

Most of the dynamic models of bridge structures are designed according to these criteria and excellent response predictions have been achieved (e.g., Ref. 4). In models of building structures these criteria need to be applied when the mass distribution in walls, floor slabs and frame elements has a significant effect on the dynamic response.

When a correct simulation of mass distribution in space is not essential, the design of models can be simplified considerably. It is then often possible to concentrate structural and non-structural masses at specific locations such as floor levels in buildings. The scaling law for these lumped masses is

$$M_r = E_r \ell_r^2 \qquad (8)$$

where E_r is the stiffness scale for the structural material. This equation is derived by multiplying $(E/\rho)_r = \ell_r$ by ℓ_r^{-3}. By using a Rayleigh approach it is also possible, in an approximate manner, to include in the lumped masses distributed between the floor levels.

Lumped mass models of building structures are being used extensively for seismic response studies (5 to 7). They are feasible for a simulation of the dynamic response of two-dimensional systems (frames and walls) but cannot account for many three-dimensional effects such as floor slab - frame interaction. If this interaction is important, discrete masses must be distributed over the floor slabs in a manner that permits simulation of gravitational and inertial effects.

The similitude requirements summarized in column (1) of Table 1 apply to models with artificial mass simulation with the exception of $(E/\rho)_r = \ell_r$ which is replaced by Eqs. (6) to (8). For completeness, the requirements are repeated in column (2) of this table.

Ultimate Strength Modeling Without Simulation Of Gravity Forces

For certain kinds of structural configurations the stresses induced by gravitational effects may be very small compared to stress histories generated by dynamic effects. In this case the gravitational acceleration g can be omitted from Eqs. (4) and (5) which eliminates the constraint $g_r = 1$ and permits the selection of a third independent model scale. If the material density ρ is chosen for this purpose, the similitude requirements shown in column (3) of Table 1 can be derived. In addition, all masses which are not part of the structural system have to be scaled according to $M_r = E_r \ell_r^3$. If prototype material is used (i.e., $E_r = \rho_r = 1$), all scale factors in this column can be expressed in terms of ℓ_r. Models of this type are easier to build since no artificial mass simulation is needed. However, the large scaling requirements for time ($t_r = \ell_r$) and acceleration ($a_r = \ell_r^{-1}$) may cause problems in measurement accuracy and in reproducing the dynamic excitation, and will amplify material strain rate effects.

In post-elastic tests, gravity effects on stress histories are always present. It is up to the model analyst to decide whether or not the errors introduced by neglecting gravity effects are acceptable. If the errors are too large, true model distortion is evident and either the prediction equation or dimensionless products which depend on a/g have to be modified to compensate for the error. When material and geometric nonlinearities are involved in the problem this will be an extremely difficult task.

Structures that may be suitable for model testing without simulation of gravity forces are, for instance, containment vessels in which the stresses induced by gravity are small compared to stress histories caused by internal pressurization and dynamic effects such as impact or earthquake ground motion. Models of this type may also be useful for dynamic response studies on slender shear walls provided the walls and boundary elements attract small gravity forces.

Linear Elastic Models

When a model test is concerned only with linear elastic behavior, gravitational effects can usually be decoupled from dynamic effects. Thus, dynamic response characteristics can be simulated in the model without regard to the scaling of the gravitational acceleration, and the similitude requirements in column (3) of Table 1 can again be applied. This makes model design for linear elastic tests a simpler task since the only material constraints are that of a linear elastic range characterized by E and, when needed, similitude of Poisson's ratio ν and material damping properties. Secondary effects caused by geometric nonlinearities are, however, not simulated properly in this type of model .

Ultimate Strength Models With Strain Distortion

In true ultimate strength models a requirement for similitude is that all dimensionless quantities must be equal in model and prototype. Thus, strains must be equal ($\varepsilon_r = 1$) and the stress-strain curves of model and prototype materials should be identical except for a constant multiplier E_r in the stress direction (see Fig. 1a). It may sometimes be impossible to fulfill this stringent similitude requirement for model materials and strain distortion will have to be accepted. If the strain distortion is constant, that is, if it can be described by a single parameter ε_r (see Fig. 1b), it appears possible to account for this distortion by modifying the Π-factors shown in Eq. (5).

Rocha [8] suggested to introduce strain as an additional basic quantity in the dimensional matrix and to assign a dimensional exponent of one to all quantities dependent on ε. Doing this will result in the similitude requirements shown in column (4) of Table 1. Since displacement and length are scaled differently, secondary effects caused by geometric nonlinearities are not simulated properly in models with strain distortion. Also, the assumption of linear dependence of physical quantities on strain may need further verification for dynamic problems.

If steel is used as reinforcement for strain distorted models, the reinforcement area should be scaled by $E_r \ell_r^2$ and the yield ε_r. This is necessary to accomplish compatibility in strain similitude between steel and concrete.

The types of models discussed here by no means exhaust the possibilities of dynamic model studies. For specific problems one may take advantage of available knowledge on the physical phenomenon under study which may permit model design with various kinds of distortions. It is then left to the ingenuity of the model designer to minimize the effects of distortions and to evaluate their influence on the response prediction.

ADDITIONAL SIMILITUDE CONSIDERATIONS

The specific purpose of a model study will dictate the degree to which the aforementioned similitude requirements have to be fulfilled and to which additional considerations will enter the model experiment. Such considerations may include simulation of initial and boundary conditions, the dynamic loading mechanism, the time, temperature and size dependence of material properties and their effects on the dynamic response and the failure modes.

The intent of the following brief discussion is to give an overview of simulation problems and solution techniques rather than be specific on any one aspect. For detailed information the reader is referred to the extensive literature in this field of which Refs. 10 to 14 are representative examples.

Simulation Of Initial Conditions

In theory, simulation of initial conditions will require the tracing of all time history events affecting the prototype, from the time of construction to the time at which the dynamic event takes place. These events will include, amongst others, construction sequence, curing methods, time dependent material effects (creep, shrinkage, increase in compressive strength of concrete), as well as damage accumulation due to previous events.

Modeling of combined creep and shrinkage effects is virtually an impossible task because of incompatibilities in time scaling. Simulation of creep requires equal time in model and prototype ($t_r = 1$) while simulation of shrinkage requires large time scaling. Creep and shrinkage depend strongly also on the size and grading of the aggregates which are not reproduced at model scales. Considering these problems it is usually a futile exercise to attempt modeling of creep and shrinkage phenomena, particularly since they often have a reasonably small influence on the ultimate strength behavior under cyclic loading. It appears to be more appropriate to control the initial conditions in the model experiment by minimizing shrinkage and creep through sealing of the model structure (for instance, by applying coats of shellac) and avoiding long-time preloading prior to the dynamic test.

Simulation of stresses and deformations introduced during construction is an equally difficult task. Unless the self-weight of the model concrete is artificially augmented during construction and creep and shrinkage are properly simulated, the effects of construction sequence are not represented in the model structure. If the selfweight of the model concrete is not augmented, the induced selfweight stresses will usually be small ($\sigma_r^{grav} \propto \ell_r$) and will remain up to the time of testing if creep and shrinkage are minimized as was suggested previously. Well defined initial conditions will then exist in the model which facilitates interpretation of test results.

The possibility of simulating damage accumulation due to a series of time history events exists in physical models to a larger degree than in analytical models. Damage due to long-time events can hardly be simulated because of inadequate modeling of creep and shrinkage, but it is possible to subject a model structure to those past transient events that may affect the initial conditions and consequently the response in the dynamic study.

Simulation Of Boundary Conditions

In dynamic model studies a proper simulation of boundary conditions may be even more essential than in static model studies. Taking seismic response studies on shake tables as an example, supporting the model structure rigidly on the table will not allow for the rocking and translational motion at the