From the results of our studies we have derived the equation for the lowest amplitude

$$A_{min} = \frac{4}{\sqrt{n_s^3}} \text{ , cm} \qquad (18)$$

and the equation for the coefficient of damping

$$\beta_u = m_1 \text{ (0.06644 log K - 0.0004 } n_s + 0.06356 \qquad (19)$$

The relation has been derived for n_s = 25-200 sec^{-1} and for the range of consistency measured by a Vebe apparatus K = 5-40 °Vebe. The coefficient m_1 depends on the shape of the grains of aggregate, with values given in Table 4-7.

TABLE 4-7 COEFFICIENTS OF COMPACTIBILITY, m_1

Values of m_1	Description
1.0	River aggregate, well rounded, l/d < 1.5
1.25	River aggregate, l/d > 1.5 < 3.0
1.75	River sand, crushed coarse aggregate
2.00	Crushed sand and coarse aggregate, l/d (coarse aggregate) < 3.0
2.25	Crushed sand and coarse aggregate, l/d (of coarse aggregate) > 3.0 < 5.0

Then

$$I_{min} = 16 \, e^h \cdot m_1 \text{(0.06644 log K-0.0004 } n_s + 0.06356) \qquad (20)$$

The lowest frequency necessary to make grains of a certain diameter D vibrate in the first phase of compaction, when the grains are still free to vibrate, was found by the equation

$$n_s = \frac{460}{\sqrt{D \, \gamma_K}} \qquad (21)$$

where γ_K = specific weight of aggregate, g/cm^3. The intensity of vibration is further limited by its absolute value I_{min} = 16 cm^2. sec^3, as follows from Eq. (20). However, neither amplitude nor frequency must be allowed to drop below the minimum values given in Eq. (19) or (21). As the absolute minimum of frequency we have found the value

$$n_s = 25 \text{ sec}^{-1} \qquad (22)$$

When the value n_s is still lower, the typical vibrating effect vanishes and a shaking down of the mix takes place instead.

Further we have found the approximate values of the maximum amplitude depending on the consistency of the mix by the equation

$$A_{max} = \frac{b}{40\, n_s^2}, \text{ cm} \tag{23}$$

where $b = 5 \times 10^4$ in slightly cohesive mixes with cement amount $C < 250$ kg/m^3, and $b = 10 \times 10^4$ in cohesive plastic mixes with $C > 350$ kg/m^3. Vibration intensity can be related to the parameters of the vibration table by means of a simple equation (Code 353), i.e.

$$I = \frac{Q^2}{M^2\, 16\, \pi^4\, n_s}, \text{ cm}^2 \cdot \text{sec}^{-3} \tag{24}$$

where M = vibrating mass (Code 352), kg . sec^2. cm^{-1}, and Q = centrifugal force (Code 353), kp. The determination of further relations is a matter of simple mathematical calculations and that is why it is not mentioned here.

Only the significance of Eq. (24) is emphasized. From this equation it can be seen that the vibration intensity changes in direct proportion to the power capacity of the electric motor (or its centrifugal force), and in an inverse proportion to the weight of the vibrated element and the frequency. It is therefore wrong to assume that a vibration table of a certain type should always have the same vibration intensity.

RELATION BETWEEN VIBRATION INTENSITY I, CONSISTENCY K, AND REQUIRED TIME OF VIBRATION T

One of the principal problems of the standard production of precast elements is to achieve a uniform quality and this again depends on the homogenous compaction of concrete. The proper relation between the consistency of the mix K, the intensity of vibration I, and the time of vibration T has to be found:

$$I = f_1(K,T), \qquad K = f_2(I,T), \qquad T = f_3(I,K)$$

By evaluating a whole series of tests, the following relations have been found:

$$I = e^{2.3026\varphi}, \text{ cm}^2 \cdot \text{sec}^{-3}$$

where

$$\varphi = 3.50 - 12.5\sqrt{0.16^{\frac{T}{K}} - 0.0144}$$

and we denoted this relation as function Φ. In solving these equations for T or K these quantities can be expressed explicitly

$$T = K\left[0.0366\ (\log I)^2 - 0.2879 \log I + 0.5843\right],\ \text{minutes} \quad (26)$$

or

$$K = \frac{T}{0.0366\ (\log I)^2 - 0.2879 \log I + 0.5843},\ {}^{\circ}\text{Vebe} \quad (27)$$

The equations hold only within certain limits; for $T_{max} = 30$ min, $K = 5\text{-}40$ $^{\circ}$Vebe, $I_{min} = 16\ cm^2 . sec^{-3}$, it is in the range of practical values.

We shall denote the product of the vibration intensity and the time necessary for a thorough compaction of the mix, I T, as a "dose," i.e. the vibration output or amount of the energy imparted to the mix. We have pointed out above, that the intensity of vibration has both a qualitative and quantitative aspect which is given partly by limit values, partly by the influence of the individual components of the vibration intensity (i.e. of the frequency [Eq. (14)] and the amplitude).

This vibration having a quantitative and qualitative aspect may be considered as having a material substance and as such it can be used in doses like the other components of the mix.

From the above relations it follows that a given concrete mix, having a constant consistency, has to be supplied with an adequate quantity of compacting energy, resulting from function Φ if it is to attain a perfect compaction.

Similarly, when changing the consistency of the mix by changing its composition we must also change the required dose of compacting energy, just as it is necessary to change the amount of cement, if we want to change the consistency while keeping the strength of concrete constant.

From Eq. (27) we can further see that the consistency K varies with the changing vibration intensity I, even if the vibration time T is constant. That means that the change of intensity is of the same importance as the change of consistency, and thus the intensity of vibration becomes a rheological factor.

Function Φ, in this simplified form (instead of the general rheological characteristic, we consider only the consistency K) may become a theoretical basis for the control of the compaction process, since fluctuations in the consistency can be compensated for by a change either of intensity or of time of vibration.

PART II

Calculation of Example I is carried out for the sake of illustration, without applying an automatic computer. The calculation procedure is in agreement with the working scheme (Figure 4-1) and flow chart (Figure 4-2). The derived parameters are computed successively

(301) $f_{cr} = \frac{f'_c}{1 - t\,v} = \frac{330}{1 - 3 \times 0.083} = 434\ \text{kp/cm}^2$

(307) $w' = \frac{a_K \times \varkappa_c}{f_{cr} + 0.5 a_K \varkappa_c} = \frac{0.5 \times 488}{434 + 0.5 \times 0.5 \times 488} = 0.439$

(316) $n_1 = 0.20 + 0.5 \times 0.265 \log(10 + 10 \times 0.265) = 0.335$

(304a) $d_1 + d_2 = 0.1 + 5 = 5.1\ \text{mm}$

(304b) $d_2 + d_3 = 5 + 20 = 25\ \text{mm}$

(319a) $x_1 = 2.685 + 3.322 \log 5.1 = 5.010$

(319b) $x_2 = 2.685 + 3.322 \log 25 = 7.329$

(320a) $n_{k_1} = 1.1 \left\{ 0.005 + 0.008(10 - 5.01) + e^{-[2.20 + 0.25 \times 5.01 + 0.05 \times 5.01^2]} \right\} = 0.0594$

(320b) $n_{k_2} = 1.1 \left\{ 0.005 + 0.008(10 - 7.329) + e^{-[2.20 + 0.25 \times 7.329 + 0.05 \times 7.329^2]} \right\} = 0.030$

(312) $\pi' = 1 - \left[\left(0.012 \frac{0.006}{3.322 \log 20 + 0.678} \right) (83 - 10 \times 0.439) \left(1 - \frac{5}{20} \right) \right] = 0.36$

(313) $y_z = \frac{\log 0.36}{\log \frac{5}{20}} = 0.737$

(314a) $\zeta_1 = \left(\frac{5}{20} \right)^{0.737} = 0.36$ (in our case, when the first fraction is identical with the fraction of sand then $\zeta_1 = \pi'$)

(314b) $\zeta_2 = \left(\frac{20}{20} \right)^{0.737} - \zeta_1 = 1.00 - 0.36 = 0.64$

(323) $Y = 0.36 \times 0.0594 + 0.64 \times 0.030 = 0.04058$

(325) $\Pi = 0.439 \times 3.03 + (1 + 0.439 \times 3.03) 0.04058 - 0.335 = 1.0898$

(326) $\Gamma = 173 - 66.5 \log 20 - 46.5 \log \frac{20}{15} = 80.1$

(351) $\beta_u = 1.0\ (0.0644 \log 20 - 0.0004 \times 50 + 0.06356) = 0.127$

(352) $M = \frac{850 + 3470}{981} = 4.404\ \text{kg} \cdot \text{cm}^{-1} \cdot \text{sec}^2$

(353) $A = \frac{13,000}{4.404 \times 39.478 \times 50^2} = 0.0299\ \text{cm}$

(354) $I = 0.0299^2 \times 50^3 = 111.75\ \text{cm}^2 \cdot \text{sec}^{-3}$

(359) $J = 0.0366\,(\log 111.75)^2 - 0.2879 \log 111.75 + 0.5843 = 0.14819$

(361) $T = 20 \times 0.14819 = 2.964 = 3$ minutes

Resulting parameters

(401) $c = \dfrac{1000 \times 0.04058 + 80.1}{1.0898} = 110.7$

(402) $V_n = \dfrac{0.439 \times 3.03\,(1000 \times 0.04058 + 80.1)}{1.0898} = 147.3$

(403) $k = \dfrac{1000\,(0.439 \times 3.03 - 0.335) - 80.1\,(1 + 0.439 \times 3.03)}{1.0898} = 741.9$

(411a) $z_1 = 741.9 \times 0.36 = 267.08 \doteq 267.1$

(411b) $z_2 = 741.9 \times 0.64 = 474.81 \doteq 474.8$

(419) $z_1 + z_2 = 267.08 + 474.81 = 741.89 = k$ in Eq. (403)

(404) $C = 110.7 \times 3.03 \doteq 335\ \text{kg/m}^3$

(405) $V_n = 335 \times 0.439 \doteq 147.1$ compare (402)

(412a) $Z_1 = 267.08 \times 2.65 = 707.76\ \text{kg/m}^3$

(412b) $Z_2 = 474.81 \times 2.72 = 1291.48\ \text{kg/m}^3$

(415) $Z_1 + Z_2 = 1998.24\ \text{kg/m}^3$

(420a) $Z_1^o = 707.76\,[1 + 0.01\,(4.5 - 0.5)] = 736.07\ \text{kg/m}^3$

(420b) $Z_2^o = 1291.48\,[1 + 0.01\,(1.2 - 0.5)] = 1300.52\ \text{kg/m}^3$

(421) $Z_1^o + Z_o^o = 736.07 + 1300.52 = 2036.59\ \text{kg/m}^3$

(416) $\bar{N}_o = \dfrac{0.5 \times 1998 \times 24}{1998.24} = 0.5$ percent

(417) $\bar{N}_s = \dfrac{4.5 \times 736.07 + 1.2 \times 1291.48}{1998.24} = 2.433$ percent

(418) $V_B = 147.3 + 0.01\,(0.5 - 2.433) \times 1998.24 = 108.67\ \text{l/m}^3$

Check calculation

(501) $c + V_n + z_1 + z_2 = 110.7 + 147.3 + 267.1 + 474.8 = 999.9 \doteq 1000$

(502) $\delta_i = 335 + 108.7 + 2036.6 = 2480.3\ \text{kg/m}^3$

(506) $\tau = \dfrac{110.7 \text{x} 0.335 + 267.1 \text{x} 0.0594 + 474.8 \text{x} 0.030 + 173 - 46.5 \log\frac{20}{15} - 147.3}{20}$

$= 4.3545$

(507) $K = 2^{4.3545} = 20.45 \doteq 20\ ^o\text{Vebe}$

(551) $I_{min} = 16 \times e^{12 \times 0.127} = 73.152 < 111.75\ \text{cm}^2 \cdot \text{sec}^{-3}$ (354)

(552) $A_o^{min} = \dfrac{4}{\sqrt{50^3}} \times e^{\frac{12}{2} \times 0.127} = 0.02418\ \text{cm} < 0.0299\ \text{cm}$ (353)

The resulting parameters of the mix composition (rounded off) are:

Cement $C = 335\ kg/m^3$	564.7	lb per cu yd
Fine fraction $Z_1^o = 736\ kg/m^3$ (moisture included)	1241	lb per cu yd
Coarse fraction $Z_2^o = 1300\ kg/m^3$ (moisture included)	2191.4	lb per cu yd
Mix water $V_B = 108.7\ kg/m^3$ (corrected with regard to moisture of aggregate)	22.02	gal. per cu yd
Water, net weight $V_n = 147.3,\ l/m^3$	29.75	gal. per cu yd
Time of vibration T = 3 minutes		

The calculation of Example II was carried out by means of an LGP 30 automatic computer following the block diagram of Figure 4-3. The determining parameters and the material characteristics are given in Table 4-8.

The derived and resulting parameters are indicated in Table 4-9, which shows the code of the relation contained in the text; the mantissa and the exponent; then, the symbol of the computed value; next, the unit of measurement; and at last, the rounded-off values of the parameters. In the table the resulting parameters are computed in the metric system, and mix proportion data are also converted to pounds per cubic yard.

It is evident that the whole program can immediately be applied for an arbitrary system of weights and measures provided that the determining parameters and the material characteristics have already been converted to metric units. When the calculation process has been finished, the results may then be transposed into the chosen system of measures and weights.

Further it is obvious that the adjustment of the mathematical relations to an arbitrary system of measures and weights does not cause any difficulties.

TABLE 4-8 DETERMINING PARAMETERS AND MATERIAL CHARACTERISTICS

Code	Parameter	Value	Unit	Code	Parameter	Value	Unit
101	f'_c	330	kp/cm^2	201	κ_c	493	kp/cm^2
104	K	30	^{o}Vebe	203	γ_c	3.12	g/cm^3
108	t_m	20	^{o}C	204	B_p	0.2731	m^2/g
109	1-t v	0.75	-	205	a_K	0.5	-
110	T_n	28	days	208a	γ_1	2.58	g/cm^3
				210	b	1.03	-
				211a	N_s^1	0.8	percent
				212a	N_o^1	0.0	percent
				213a	d_1	0.125	mm
				213b	d_2	5.00	mm
				213c	d_3	15.00	mm
				213d	d_4	30.00	mm
				214a	γ_2	2.624	g/cm^3
				214b	γ_3	2.664	g/cm^3
				216	m_K	78	-
				218a	N_s^2	0.3	percent
				218b	N_s^3	0.2	percent
				219a	N_o^2	0.0	percent
				219b	N_o^3	0.0	percent
				220	D_{max}	30	mm

TABLE 4-9 CALCULATION OF PRELIMINARY DIRECTIVE VALUES OF THE PROPORTION OF A CONCRETE MIX PERFORMED ON AN LGP-30 COMPUTER

Code	Mantissa	Exponent	Symbol	Units	Technical transcription
(301)	4400000	03		kp/cm^2	440
(304a)	5125000	01	$d_1 + d_2$	mm	5.125
(304b)	2000000	02	$d_2 + d_3$	mm	20.0
(304c)	4500000	02	$d_3 + d_4$	mm	45.0
(305)	1003588	01	x_m	coefficient	1.0
(104)	3000000	02	K	°Vebe	30
(306)	4249062	00	w	coefficient	0.425
(310)	5925826	00	σ'	ratio	0.592
(312)	4074174	00	$\pi \equiv \zeta_1$	ratio	0.407
(316)	3508708	00	n_1	coefficient	0.351
(319a)	5042632	01	x_1	coefficient	5.043
(319b)	7007075	01	x_2	coefficient	7.007
(319c)	8177040	01	x_3	coefficient	8.177
(320a)	5507122	01-	n_{k1}	coefficient	0.055
(320b)	3151163	01-	n_{k2}	coefficient	0.032
(320c)	2069316	01-	n_{k3}	coefficient	0.021
(313)	5011370	00	y_z	exponent	0.501
(314b)	2991323	00	ζ_2	coefficient	0.299
(314c)	2934502	00	ζ_3	coefficient	0.293
(323)	3793553	01-	Y	function	0.0379
(325)	1063063	01	Π	function	1.063
(326)	6077213	02	Γ	function	60.77
(401)	9285210	02	c	$1/m^3$	92.85
(404)	2896985	03	C	kg/m^3	290 (488.9 lb/cu yd)
(405)	1230947	03	V_n	$1/m^3$	123
(406)	7840532	03	k	$1/m^3$	784
(411a)	3194369	03	z_1	$1/m^3$	319
(411b+ 411c)	4646162	03	$z_2 + z_3$	$1/m^3$	465
(411b)	2345356	03	z_2	$1/m^3$	235
(411c)	2300806	03	z_3	$1/m^3$	230
(412a)	8241472	03	Z_1	kg/m^3	824 (1389.0 lb/cu yd)
(412b)	6154215	03	Z_2	kg/m^3	615 (1036.7 lb/cu yd)
(412c)	6129347	03	Z_3	kg/m^3	613 (1033.3 lb/cu yd)
(415)	2052503	04	K_m	kg/m^3	2052
(416)	4709035	01-	N_s	percent	0.47
(417)	0000000	00	N_o	percent	0.0
(418)	1240612	03	V_B	$1/m^3$	124 (209 lb/cu yd)
(501)	1000000	04	control	$1/m^3$	1000
(506)	4911489	01	τ	exponent	4.911
(507)	3009578	02	K	°Vebe	30
(502)	2465296	04	δ_1	kg/m^3	2465 (4155.9 lb/cu yd)

PAPER SP 16-5 Thin shell analysis formulated by finite elements is described as a general procedure well suited to computers. Analysis of the individual elements can conveniently be done by numerical integration and examples are described for barrels, domes, and shell walls. Confidence in these analyses are established by studying for all shells the maximum integration length and for domes the required ficticious crown opening. The superposition method of analysis is described for continuous folded plates and finally a brief discussion is given of extensions in the numerical integration of barrels to include the inelastic behavior of reinforced concrete thin shells, particularly above working loads.

Computers and Thin Shell Analysis

By DAVID P. BILLINGTON

THE PURPOSE OF THIS PAPER is to discuss some methods of analysis for thin shell concrete structures which have been introduced specifically with computer solutions in mind. While most of this discussion concerns elastic analysis the paper concludes with some considerations of analyses formulated to treat the inelastic behavior of thin shells built of concrete.

Because thin shells require more complex analyses than other structural systems it is natural that research in analysis should emphasize computer applications. Furthermore since the design of thin shell concrete structures must often be based upon an analysis radically simplified from a rigorous mathematical theory, one goal of research is to discover the range of validity of the various simplifications. In the research directed to the development of more general methods of analysis three aspects have been central to the work described here:

Formulation -- The shell equations are formulated directly with the computer in mind. Analytic procedures developed before the computer was generally available are usually not followed. Rather the formulation proceeds from the basic equations themselves and is done with the particular potentialities of the computer directly in mind right from the beginning.

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Programming -- The emphasis is placed upon the use of existing computer library subroutines, rather than upon extensive and sophisticated programming.

Confidence -- When new types of formulations are used and when existing subroutines are employed for computation it is essential that the confidence in the accuracy of the results be gained by some analysis of the validity of the methods used over the desired range. In many of the newer methods of shell anslysis the question of the accuracy is crucial and usually necessitates some analytically defined limits for the range of validity of the particular formulation used.

With these ideas in mind a series of analyses is discussed, several are described in detail, and the paper concludes with a discussion of the use of some of these methods of analysis for treating the inelastic behavior of thin shell concrete structures.

ANALYSIS BY FINITE ELEMENTS

The method of finite elements was first introduced by R. Clough[1] for the solution of two-dimensional diaphragms and has been since extended to dams, domes, and many other systems. This powerful method is general enough to include all the procedures discussed in this section and is used here to emphasize the unity which underlies procedures which have previously been given different names. In fact, Popov et al.[2] have already used the term finite element for thin shell analyses in the sense intended for this paper.

Two general types of thin shells are discussed here: rotational and translational. In general the finite element method consists of dividing the smooth shell surface into a series of small elements--rings in the case of rotational shells, and strips