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Should Design Codes Consider Fracture Mechanics Size Effect?

by Zdeněk P. Bažant

Synopsis. The paper reviews recent theoretical and experimental results on the size effect in brittle failures of reinforced concrete structures caused by the release of stored energy. After summarizing the size effect law and explaining the novel concept of a brittleness number, the results of recent Northwestern University tests of diagonal shear failure, punching shear failure, torsional failure and pullout failure are discussed. These results, which were obtained on geometrically similar specimens with a broad range of sizes are found to be in excellent agreement with the theoretical size effect law. The experimental evidence is much stronger than that which was previously obtained by analyzing a large amount of test results from the literature, which were not obtained on geometrically similar specimens and were limited to a narrow size range. It is also pointed out that the test data on diagonal shear disagree with the classical Weibull-type theory of size effect, thus strengthening the theoretical argument against using this theory for the size effect in concrete structures whose maximum load is much larger than the cracking initiation load. The test results indicate that the presently considered fracture mechanics size effect ought to be incorporated into the formulas for the contribution of concrete to the ultimate load capacity in brittle failures of concrete structures. It is shown that such formulas can be based on the brittleness number. For any given structure shape, this number can be determined from size effect tests. However, prediction of this number without such test data will require some further research.

Keywords: Concrete structures; diagonal tension; failure; fracture mechanics; pullout tests; punching shear; shear properties; size effect; standards; structural design; torsion

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1 Introduction

It has long been known that ultimate loads of concrete structures exhibit size effect. The classical explanation has been Weibull's weakest-link theory which takes into account the random nature of concrete strength [1,2,3,4,5]. However, for reasons given elsewhere [6] and briefly explained in the Appendix, it now appears that the statistical theory does not suffice to describe the essence of the size effect observed in brittle failures of reinforced concrete structures and plays only a secondary role. The main mechanism of the size effect in this type of failure is deterministic rather than statistical, and is due to the release of the stored energy of the structure into the front of the cracking zone or fracture. This phenomenon is properly described by fracture mechanics in its recently developed nonlinear formulation which takes into account the distributed nature of cracking at the fracture front.

The purpose of this review paper is to summarize the existing evidence and also present some recent experimental results obtained at Northwestern University.

2 Mathematical Description of Size Effect

The size effect is defined by comparing the ultimate loads (maximum loads), P_u, of geometrically similar structures of different sizes. This is done in terms of the nominal stress σ_N at failure. For two-dimensional similarity (e.g., panels), $\sigma_N = c_n P_u/bd$, and for three-dimensional similarity (e.g., cylinders), $\sigma_N = c_n P_u/d^2$. Here b = thickness of a two-dimensional structure; d = characteristic dimension (size), which may be defined as any dimension of the structure, e.g., the depth of a beam or its span, since only the relative values of σ_N matter; and c_n = chosen dimensionless coefficient introduced for convenience. One may either set $c_n = 1$ or use c_n to make σ_N coincide with some convenient stress formula. E.g., for a simply supported beam of span L and a rectangular cross section of depth H, with load P at midspan, one may set $d = H$ and $c_n = 3L/2H$, in which case $\sigma_N = 3PL/2bH^2$ = maximum elastic bending stress (c_n is constant because L/H is constant for geometrically similar structures); or one may set $d = L$ and $c_n = 3L^2/2H^2$, with the same result for σ_N.

When the σ_N- values for geometrically similar structures of different sizes are the same, one says that there is no size effect. The size effect represents a dependence of σ_N on the structure size (characteristic dimension), d.

According to plastic limit analysis, as well as elastic analysis with allowable stress or any theory that uses a failure criterion in terms of stresses or strains, σ_N is independent of the structure size. This can be illustrated, e.g., by the elastic and plastic formulas for the strength of beams in bending, shear or torsion [7].

Another theory of failure, conceived by Griffith [8] and introduced to concrete by Kaplan [9], is fracture mechanics. It was Reinhardt [10,11] who proposed that fracture mechanics should be used to describe the size effect in concrete structures, particularly in diagonal shear failure. He also showed that the size effect of classical, linear elastic fracture mechanics agrees reasonably well with some test results, although later it was found that nonlinear fracture mechanics is necessary in general.

In the linear form of fracture mechanics, in which all the fracture process is assumed to be happening at a point—the crack tip—the size effect is the strongest possible. In the plot of log σ_N vs. log d, it is described (regardless of the structure shape) by an inclined straight line of slope $-1/2$ (Fig. 1), provided that the cracks at the moment of failure of geometrically similar structures of different sizes are also similar. The reason for stipulating this condition (which has been shown from tests [12,13,14,15,16,17] to be usually satisfied) will be briefly explained after Eq.1.

Concrete structures in reality exhibit a transitional behavior between the size effect of strength or yield criteria (i.e., no size effect), represented in Fig. 1 by a horizontal line, and the size effect of linear elastic fracture mechanics, represented by the straight line asymptotic of slope $-1/2$; see the curve in Fig. 1. This size effect is generally ignored by the current design codes, but recent tests [12,13,14,15,16], as well as Eq.1, show it to be very strong, and thus important.

The aforementioned transitional size effect can be most simply explained by considering uniformly stressed rectangular panels of different sizes d, loaded by uniform distributed load σ_N, as shown in Fig. 2. Each panel is assumed to have a weak spot in the middle of the left side, from which the fracture originates. For a brittle heterogeneous material such as concrete, it is important to take into account a relatively large zone of distributed cracking at the fracture front. The size of this zone is not proportional to the structure size but is approximately related to the maximum aggregate size. In the simplest approximation, it may be assumed that the width, h, of the cracking band at the fracture front is approximately constant, independent of the structure size (when similar structures made from the *same* concrete are compared). Likewise, it may normally be assumed that, at maximum load, the length of the fracture, a, is proportional to dimension d of the structure, i.e., a/d = constant. (This is supported by many of the brittle failures of reinforced concrete structures, as well as by finite element fracture studies.)

Formation of a fracture with crack band of thickness h and length a may be imagined to release the strain energy of density $\sigma_N^2/2E$ from the cross-hatched area in Fig. 2 (E = elastic modulus of concrete). When the fracture extends by Δa, the additional strain energy that is released into the fracture front comes from the densely cross-hatched strip of horizontal dimension Δa. Obviously, the larger the structure, the larger is the area of the cross-hatched strip, which is given by $\Delta A = h\Delta a + 2ka\Delta a$ where k = slope in Fig. 2 = empirical constant depending on the structure shape (the k-value can be deduced from test results or finite element analysis; for the panel, $k = \pi/2$, but the value of k does not matter for the present argument, only the fact it is a size-independent constant). Now it is crucial to realize that in a larger structure the energy that is released into a certain small extension Δa of the fracture is larger if the σ_N–value is the same because it comes from a zone of a larger volume. Since the energy dissipated by fracture per unit area of the fracture plane is approximately constant (being equal to the fractaure energy, G_f, which is a material property), the value of σ_N for a larger structure must be less so that the total energy release from a zone of a larger volume would remain the same. Hence the size effect.

The strain energy released from the aforementioned densely cross-hatched strip is $\Delta W = b(h\Delta a + 2ka\Delta a)\sigma_N^2/2E$ where b = panel thickness. Setting $\Delta W = G_f b\Delta a$ = dissipated energy, one obtains $\sigma_N^2[h + 2k(a/d)d] = 2EG_f$. Solving for σ_N, one can bring the resulting expression to the form of the size effect law [7]:

$$\sigma_N = Bf_t'(1+\beta)^{-\frac{1}{2}}, \qquad \beta = d/d_0 \tag{1}$$

in which the following notations have been made: $B = (2EG_f/hf_t'^2)^{1/2}, d_0 = hd/2ka$, and f_t', representing the direct tensile strength of concrete, is introduced to make B nondimensional. The ratio β is called the brittleness number of the structure, for reasons explained later. Now it is important to note that parameters B and d_0 are size-independent, i.e., constant, because d/a is constant if there is geometric similarity (see hypothesis 3 below), and h is also approximately size-independent, as already mentioned.

It must be emphasized that Eq.1 is only approximate. But its accuracy is sufficient for a rather broad size range—from experience, up to about 1:20, which is adequate for most practical purposes. For a still broader size range, a more complicated formula would nevertheless be required.

For small enough structures (compared to d_0), i.e., $d << d_0$, Eq.1 yields $\sigma_N = Bf_t'$ = constant, which means that the size effect disappears (see the horizontal asymptote in Fig. 1). The plastic limit analysis or elastic allowable stress design is then valid. This has been the case for most laboratory testing so far. For $d >> d_0$, the fracture process zone size is negligible compared to the structure size, which is the case of linear elastic fracture mechanics. Eq.1 reduces in this case to $\sigma_N = Bf_t'\beta^{-1/2}$ or $\log \sigma_N = -\frac{1}{2}\log d +$ const., which gives in Fig. 1 a straight-line asymptote of slope $-\frac{1}{2}$. Thus it is clear that Eq.1 gives a smooth transition between these two asymptotic cases. The intersection point of the asymptotes is obtained by setting $Bf_t' = Bf_t'\beta^{-1/2}$, which yields $\beta = 1$ or $d = d_0$ (Fig. 1).

Since from some viewpoints the length of the distributed cracking zone at the fracture front is more important than the width, it is interesting to note that a derivation of Eq.1 similar to that given above, with the same result [7,18] can be made for a sharp line crack having a fracture process zone of constant length at the front. For more complicated structural geometries, the foregoing type of reasoning gets difficult. However, Eq.1 can be derived generally by dimensional analysis and similitude arguments [7,19]. This general derivation rests on two basic hypotheses: (1) the propagation of a fracture or crack band requires an approximately constant energy supply per unit length and width of fracture, and (2) the potential energy released by fracture from the structure is a function of both (a) the length of the fracture and (b) the area of the cracking zone (fracture process zone) at the fracture front. If the potential energy release is a function of only the fracture length, the size effect of linear elastic fracture mechanics ensues, and if it is a function of only the cracking area, there is no deterministic size effect.

The transitional size effect curve in Fig. 1 is also obtained by numerical models of the microstructure, such as the random particle model. In this model, a system of aggregate particles is generated randomly and each large aggregate particle is considered as a finite element interacting with its neighbors through a contact element representing the contact zone [20]. Furthermore, nonlocal finite element models in which localization of cracking is restricted to a zone of a certain minimum size, also exhibit the same transitional type of size effect, while ordinary finite element codes are incapable of representing it [21,22]. Finally, Eq.1 has been derived as the deterministic limit of a statistical strength theory that represents a nonlocal generalization of Weibull theory [6].

Eq.1 has been extensively verified by testing both fracture specimens [7,19, 23] and reinforced concrete structures [12,13,14,15,16,24,25,26], as well as by computer simulations of cracking propagation [20,21,22]. In the case of test specimens, similarity of the fracture shape and length is enforced by providing geometrically similar notches in specimens of different sizes. In real concrete structures, from which notches are absent, Eq.1 is applicable only under the following two additional hypotheses: (3) the failure modes (i.e., fracture shapes and lengths) of geometrically similar structures of different sizes are, at the moment of maximum load, also geometrically similar, and (4) the structure does not fail at crack initiation. From testing [12,13,14,16,24,25,26], as well as finite element (and other) computational models [20,21,22], it appears that these assumptions usually are approximately satisfied over a wide range of structure types and sizes (but exceptions exist, e.g., in brasilian split cylinder tests, caused by a change in the failure mode as the size becomes very large). A good design practice of course requires the maximum load to be much higher than the cracking initiation load (as required by hypotheses 4), and this is to some extent also enforced by design codes.

The characteristic of the failure process that gives rise to the size effect is the propagating nature of failure. In plastic limit analysis the failure is always non-propagating, simultaneous, with all the parts of the structure forming at maximum load a single-degree-of-freedom mechanism and moving simultaneously in proportion to one time-like parameter. The typical characteristic of such failures is that the load-deflection diagram, after reaching the maximum load, exhibits a horizontal plateau. It can be shown in general that when the horizontal plateau is lacking, i.e., when the load decreases after the peak with increasing deflection, the failure cannot be simultaneous but must be propagating. Propagating failures need to be generally described by fracture mechanics, *not* plastic limit analysis, and they always exhibit size effect (of the energy release type).

3 Brittleness Number

Distinctions between brittle and ductile failures have long been emphasized in concrete textbooks, however the meaning of brittleness has been left hazy, unquantified. The notion of brittleness is closely connected with the size effect. Brittleness increases with size. It can be generally shown that in small structures the load declines relatively slowly with deflection after the peak, while in a similar large structure the load-deflection curve declines steeply, and for a sufficiently large size even exhibits the so-called snapback instability in which the load-deflection curve becomes vertical, after which the failure is dynamic. Recognizing this connection, various authors, including Gogotsi et al. [27], Homeny et al. [28] for ceramics in general, and Carpinteri [29] and Hillerborg [30] for concrete in particular, proposed brittleness to be quantified by some brittleness number depending on the structure size, d. Unfortunately, Gogotsi, Homeny, Hillerborg, and Carpinteri's brittleness numbers are not independent of the structure geometry (shape) and thus cannot be used as universal characteristics of brittleness (e.g., a brittleness number equal to 2 could mean a very brittle behavior for one structure geometry and a very ductile behavior for another geometry). These numbers only allow comparing the brittleness of similar structures of different sizes.

A universal measure of brittleness is offered by the size effect law [7], although it is only approximate since the exact size effect law is not known. For $\beta >> 1$, linear fracture mechanics applies, which represents a perfectly brittle behavior. For $\beta << 1$, plastic limit analysis applies, which represents the absence of brittleness. Therefore, the ratio $\beta = d/d_O$ may be taken as a brittleness number [19,23]. According to Eq.1, the horizontal asymptote in Fig. 1 is $\sigma_N = Bf_t'$, and the inclined asymptote is $\sigma_N = Bf_t'\beta^{-1/2}$. They intersect at $\beta = 1$. Therefore, the value $d = d_0$ (or $\beta = 1$) corresponds in the size effect plot of log σ_N vs. log d to the point where the horizontal asymptote for the strength or yield criterion intersects the inclined asymptote for the linear elastic fracture mechanics (LEFM), (Fig. 1). For $\beta < 1$, the behavior is closer to plastic limit analysis, and for $\beta > 1$ it is closer to linear elastic fracture mechanics. With a practically sufficient accuracy, the nature of structure response and the type of analysis may be characterized as follows [19]:

$\beta < 0.1$	plastic limit analysis	
$0.1 \leq \beta \leq 10$	nonlinear fracture mechanics	(2)
$\beta > 10$	linear elastic fracture mechanics (LEFM)	

For $\beta < 0.1$, the horizontal asymptote deviates from Eq.1 (Fig. 1) by less than 4.7%, which is small enough to permit the use of plastic limit analysis, and for $\beta > 10$, the inclined asymptote deviates from Eq.1 also by less than 4.7%, which is small enough to permit the use of LEFM. If deviations under 2% are desired, then the nonlinear range must be expanded to $1/25 \leq \beta \leq 25$.

Let us now consider the determination of the brittleness number when size effect test data are absent. Two formulas have been derived for this purpose, based on matching the inclined asymptote of the size effect law, $\sigma_N = Bf_t'/\sqrt{\beta}$, to a solution by linear elastic fracture mechanics. They read

$$\beta = B^2 g(a_0) \frac{f_t'^2}{EG_f} \tag{3}$$

$$\beta = \frac{g(\alpha_0)}{g'(\alpha_0)} \frac{d}{c_f} \tag{4}$$

in which $g(\alpha_0)$ is the nondimensionalized energy release rate corresponding to the initial relative crack length $\alpha_0 = a_0 d$ according to linear elastic fracture mechanics (see any fracture mechanics textbook, e.g., Broek [31] or Bažant and Cedolin, Ch.12 [32]), $g'(\alpha_0)$ is its derivative, and c_f is the effective length of the fracture process zone, which is a material property (if defined for extrapolation to a specimen of infinite size).

The first formula, Eq.3 (derived in [19,23]), is more accurate for small sizes, and the second formula, Eq.4 (derived in [33]), from a modified form of Eq.1, is more accurate for large sizes. With either formula, calculations of the brittleness number necessitate knowing the shape and length of the linear elastic crack at maximum load of a very large structure. This crack is precisely defined only by extrapolation to a similar structure of infinite size (for real size structures, there is a cracking band rather than a well defined crack). For typical brittle failures of concrete structures except diagonal shear [34], it has not yet been established what shape and length this crack should have. However, once this shape and length become known, $g(\alpha_0)$ and its derivative can be easily calculated with a linearly elastic finite element program, and could also be tabulated for typical structures. For the calculation of B, the existing design formulas based on plastic limit analysis can probably be used.

Alternatively, one might also skip the determination of $g(\alpha_0)$ and develop on the basis of test data alone simple empirical formulas [34] that directly give, for various typical structure shapes, the value of the transitional size d_0 (for example in relation to the cross section dimension and the maximum aggregate size d_a). Then the brittleness number immediately results as $\beta = d/d_0$. Such a value of d_0 would not be exact, but may be accurate enough for design purposes (after all, due to the log-scale in Fig. 1, what matters is the order of magnitude of d_0; errors by factors up to 2 may be tolerable).

In view of the universality of the size effect law in Fig. 1 and the associated brittleness number, it appears that a simple adjustment can introduce the size effect into the existing code formulas based on limit analysis. It might suffice to take the existing code formula for the nominal stress due to concrete at ultimate load, v_u, and replace it by the expression:

$$v_u(1+\beta)^{-1/2} \tag{5}$$

Note, however, that for some types of failure there may exist some limit v_u^{min}, since at very large sizes there can be a transition to some nonbrittle failure mechanism (this in fact is the case for the Brazilian split-cylinder test). An empirical expression for d_0 needed for calculating β in Eq.5 has been proposed in [24,26] but a rational method to calculate d_0 [34] for most types of brittle failure still awaits development.

4 Previous Tests of Brittle Structural Failures

After the size effect law has been formulated, much effort has been devoted to comparing and validating it on the basis of the test data in the literature. The efforts were especially focused on the diagonal shear failure of reinforced concrete beams without and with stirrups [12,24,26]. The latter study included essentially all the experimental data that could be extracted from the literature, consisting of 461 beam tests. After approximately eliminating the effects of shear span, reinforcement ratio and other factors according to various known approximate formulas (as explained in these papers), the existence of a size effect has been clearly demonstrated. It was also shown that incorporation of the size effect law (Eq.1) into the existing ACI or CEB-FIP design formulas for the contribution of concrete to the ultimate strength of beams in diagonal shear brings about a distinct improvement, reducing the coefficient of variation of the deviations of the test results from the design formula.

Unfortunately, however, the results of these studies have not allowed any strong conclusions. The reason has been that the size effect data extracted from the previous tests showed enormous scatter, which was probably due mainly to the errors of the formulas used to filter out the effect of various other factors. The tests have been done at various laboratories, on various concretes, and on beams of various geometries. Most test series did not include various sizes. Those few that did ([35,36,37,38,39,40,41]), did not include a sufficiently broad range of sizes and, most seriously, did not use geometrically similar specimens and the same aggregate sizes. The same is true of the latest and largest study of the size effect in diagonal shear presented by Iguro, Shioya, Nojiri and Akiyama [42].

The lack of geometric similarity in the previous test series has been the most serious impediment against their exploitation for the present purposes. To extract information on the size effect, adjustments for all the other influencing factors had to be made first. But since the influences of those other factors are known only approximately, a considerable error is inevitably introduced by such adjustments. This causes enormous scatter, which obscures the underlying trend of the size effect [12,15,24,26].

Aside from unprestressed beams without stirrups, the previous studies of test data from the literature dealt also with prestressed beams without stirrups and with unprestressed beams with stirrups. In the latter case, the portion of the carrying capacity due to stirrups is of course free of size effect as the stirrups fail in a ductile manner. However, despite considerable scatter, the analysis of the data confirmed that, in contrast to the current design approach, the carrying capacity due to stirrups (which exhibits no size effect) is not simply additive to that due to concrete (which does exhibit a size effect). Rather, the presence of stirrups appears to have a strengthening influence on the portion of the carrying capacity due to concrete, which is of course not really surprising.

Similar problems have been encountered in an attempt to evaluate the size effect from the existing data on punching shear failures and torsional failures although, for the latter, the test results of Hsu [43], Humphreys [44] and McMullen and Daniel [45] provided at least a clear indication of the existence of a significant size effect.

To sum up, despite a clear revelation of the existence of size effect, the enormous scatter and narrow size range of the previous test data from the literature [24,26] has made it impossible to verify with such data which size effect theory is the correct one. For example, the present size effect law fits most previous data from the literature no better than a formula based on Weibull-type statistical theory. This state of affairs, for example, would permit concluding on the basis of the test data of Iguro et al. [42] that the Weibull-type theory should be acceptable for diagonal shear failures of beams, even though its use is in fact questionable from the theoretical viewpoint as already mentioned.

5 Evidence from New Tests of Geometrically Similar Structures

In view of the aforementioned limitations of the previous experimental evidence, a systematic program of new tests of brittle failures of reinforced concrete structure, focused on the size effect and strictly adhering to geometric similarity, has been carried out at Northwestern University. To keep the costs down, all the tests were done on reduced-size specimens with reduced-size aggregate (maximum sizes 3/8 or 1/4 in.). The tests included the diagonal shear failure of beams without stirrups [14] the punching shear failure of circular slabs reinforced at bottom surface [12], the torsional failures of plain and longitudinally reinforced concrete beams [25], and the pullout failure of reinforcing bars [15]. The results of these tests, whose details are given in the aforementioned articles, are shown as the data points in Figs. 3-6. The data for punching shear (Fig. 4) have the size range 1:4, and so have the data for torsion (Fig. 5a,b) and the data for pullout (Fig. 6).

The tests of diagonal shear consisted of two series. In the first series (Fig. 3a), in which the size range was 1:4 and in which the longitudinal bars were straight, it was found that the diagonal shear failure was accompanied by pullout failure of bars, marked especially for the smallest size. Therefore a second series (Fig. 3b) was conducted on beams in which the bars had right angle hooks at the ends, which prevented the pullout. The second series had the size range of 1:16. The beam specimens were similar in two dimensions, i.e., they had the same thickness for all the sizes.

From Figs. 3-6 it is clear that there is a strong size effect. It is also noteworthy that in the logarithmic scales the size effect curve does not tend to level off at large sizes, which is predicted by fracture mechanics. The optimal fits according to Eq.1 are shown as the solid curves, and it is seen that the agreement is quite good, especially in view of the inevitable statistical random scatter of concrete strength in brittle failures. While the aforementioned studies of previous test data from the literature only confirmed the existence of size effect but could not decide which formula for the size effect was the correct one, the comparisons in Figs. 3-6 can be said to support Eq.1. This is especially clear for the second series of the diagonal shear tests (Fig. 3b), by virtue of its broad size range.

The second series appears to represent the first test results that show that classical Weibull-type statistical theory does not apply to concrete structures. According to this theory, the strongest size effect in two dimensions corresponds to a straight line of slope $-1/6$, whose optimal fit to the present data is shown as the dashed line in Fig. 3b. One can see that this line clearly disagrees with the trend of the data and gives too weak a size effect for the large sizes. (This is true provided one takes the value of Weibull modulus as $m = 12$, in agreement with the results of uniaxial tensile tests, and assumes the Weibull threshold strength to be $\sigma_0 = Q$; if this threshold were larger than 0, the size effect would be even milder than that shown by the dashed line in Fig. 3b, exhibiting an approach to a horizontal asymptote, as shown by the dotted curve.)

The essential parameter that determines the intensity of the size effect is the transitional size d_0. This parameter represents a combination of a material property with the effect of structure shape. The tests in Figs. 3-6 show that its values vary greatly from one structure type to another. From the present limited results, it is not yet possible to give a good empirical formula for evaluating d_0. But parameter d_0 can also be predicted theoretically from Eq.3 or 4. In that regard, further research is desirable in order to determine the shape and length of the equivalent linearly elastic crack at maximum load, which is needed to calculate $g(\alpha_0)$ for Eq.3 or 4, or to develop empirical formulas for d_0. To obtain complete experimental verification and minimize reliance on theoretical extrapolations, geometrically similar tests of sufficient size range may have to be made with full-size aggregate, including real-size beams and slabs.