SP 183-1

# Concept and Background of Elastic Frame Analogies for Two-Way Slab Systems

by S. Simmonds

Synopsis: The justification for using elastic frame analogies to determine design moments in two-way slab systems is discussed. A brief history of two-way reinforced concrete slab design leading to the current code procedures is presented. This history includes a description of the various elastic frame analogies that have existed in past codes, the reasons for changes and the research leading to improved frame analogies. This is followed by a critical review of the Equivalent Frame Method in the current code with suggestions for improving and simplifying provisions for elastic frame analogies in future codes.

Keywords: analysis; design; elastic frames; history; reinforced concrete slabs

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## WHY AN ELASTIC FRAME ANALOGY?

Two-way slab systems are a common structural component in reinforced concrete construction. If asked how they design these slabs, many designers in North America would answer 'I use a computer program'. If pressed as to the methodology incorporated in the program they would likely respond 'elastic frame analogy'. Why an elastic frame analogy?

Traditionally, in reinforced concrete design, one uses a linear elastic theory to determine design parameters and then proportions members using an ultimate strength procedure. The justification for this apparent anomaly is that by designing for moments determined from elastic theory the amount of moment redistribution at service load conditions will be minimized thereby ensuring that serviceability requirements will generally be satisfied. Except for special cases such as deep beams or sudden changes in cross section where elastic theory is not applicable, this technique has served us well.

To apply a similar procedure to the design of two-way slab systems it is necessary to have a means of obtaining an elastic analysis. As early as 1811, Lagrange proposed an elastic theory for thin slabs which requires determining a function that will satisfy both a fourth-order differential equation and the boundary conditions. Solutions using this approach have been successful only for slabs with the simplest idealized boundary conditions, generally panels with non-deflecting boundaries. This method has been used to develop design procedures for slabs with beams between all supports. It was the need to provide a simple elastic analysis for column supported two-way slab systems that led to the concept of an elastic frame analogy.

Even today, although a number of ingenious techniques to obtain solutions for two-way systems have been proposed, for example Ang (1) and, more recently, numerical solutions based on finite element or finite difference techniques, none have proved practical for routine office use. Hence the continuing interest in elastic frame analogies.

## WHAT IS AN ELASTIC FRAME ANALOGY?

The concept behind the use of elastic frame analogies is that satisfactory values for the design moments and shears in two-way slab systems can be obtained by considering a portion of the slab-column structure to form a design frame that can be analyzed as a plane frame.

The process consists of three parts:

a) define the analogous plane frame including assigning member stiffness

b) analyze frame with appropriate loading to obtain maximum frame moments, and c) distribute frame moments laterally across the corresponding critical sections of the slab.

Frame analogies can be used for both gravity and lateral loading on slab-column structures.

The basic approach for defining the geometry of the analogous elastic plane frames has remained essentially unchanged through various codes. The structure is considered to be made up of analogous or equivalent frames centered on the column lines taken longitudinally and transversely through the building, see Fig. 1. Each frame consists of a row of columns or supports and slab-beam strips bounded laterally by the centerline of the panel on each side of the centerline of the columns or supports. Frames adjacent and parallel to an edge are bounded by that edge and the centerline of the adjacent panel. Each frame may be analyzed in its entirety, or for vertical loading each floor or roof with attached columns may be analyzed separately.

Success in applying this analogy depends on the appropriate apportioning of stiffness to the members of the frame so that the elastic analysis of the two-dimensional frame will approximate that of the non-linear three-dimensional slab-beam-column system. This problem is made more complex by a fundamental assumption in the analysis of plane frames that does not apply to slab-column systems. In a typical plane frame analysis it is assumed that at a beam-column connection all members framing into that joint undergo the same rotation as shown in Fig. 2(a). For slabs supported by columns this assumption is valid only locally at the column. Portions of the slab laterally removed from the column will rotate lesser or greater amounts depending on the geometry and loading patterns as shown in Fig. 2(b). Furthermore, actual slab systems crack even under service loading, especially near the face of the column resulting in locally reduced stiffness. To account for the differences in behavior of the actual slab-column system and the idealized plane frame, it is necessary to modify the stiffness of the frame elements. Unfortunately, the modifications required to the member stiffness for lateral loading differ from those for gravity loading.

The definition of the analogous frame, the apportioning of stiffness and the rules for the lateral distribution of design moments across the slab have evolved through successive codes. To follow this evolution, it is helpful to review the history of the development of two-way slab construction.

## 'TWO-WAY SLABS' AND 'FLAT SLABS'

Since the 1971 ACI Code, the term 'two-way slab' refers to all slab systems reinforced for flexure in more than one direction with or without beams between supports. The term 'flat slab' is not used. Prior to 1971, the term 'two-way slab' referred only to those slabs with beams between supports along all sides of each panel and the term 'flat slab' referred to slabs without beams between supports but could have column capitals and/or drop panels. The need for the distinction in earlier codes was because of the different genesis of the two slab types and the resulting differences in design rules. The elastic frame method was initially developed for two-way slabs without beams (flat slabs). In the remainder of this paper the term flat slab is used as defined above when discussing design rules prior to 1971.

## EARLY HISTORY OF SLAB DESIGN

Reinforced concrete flat slabs were invented in the sense that they were not a logical extension of elastic theory or construction practice. Credit for this invention is generally given to C. A. P. Turner who constructed his first 'mushroom slab' (a reinforced concrete slab supported on columns with flared column capitals) for the five-story C. A. Bovey Building in Minneapolis in 1906. Lacking a rational analysis, the validity of his design was verified with a load test. So successful was this slab that almost immediately competitors were constructing slabs using various proprietary methods. Since there was no generally accepted procedure for analyzing such slabs, it is not surprising that the amount of reinforcement required varied considerably from design to design. A comparison of the amounts of reinforcement required in an interior panel by six different design procedures made by McMillan (2) in 1910 showed that some designs required four times as much steel as others.

In an attempt to reconcile these difference in designs, many of the slabs that were load tested had measurements of the strains in the reinforcement. Moments in the slab were computed from these steel strains using a straight line expression. These tests did not resolve the differences in design procedures.

In 1914, Nichols (3) examined the statics of a uniformly loaded interior panel of a slab without beams with square panels extending infinitely in both directions. In his original paper, he considered only a quarter of the panel but in the closure to his paper he considered as a free body the half panel designated as A, B, C, and D in Fig. 3. From symmetry no shears or twisting moments exist on faces B, C, and D but bending moments exist on all faces. He assumed that the shear forces on face A are uniformly distributed. Denoting the sum of the moments of all vertical forces about x-x as $M_0$, yields the simplified expression

$$M_0 = \frac{WL}{8}\left(1-\frac{2c}{3L}\right)^2 \quad \text{where W is the total load on the panel.} \qquad (1)$$

The difference between this expression and the exact expression is less than 1% for values of c/L smaller than 0.3.

Nichols concluded that for equilibrium this must be the sum of the bending moments on faces C and B plus the components about x-x on face A. While this analysis does not give the actual moment at any point or even across any section, it does provide a criterion against which proposed design moments could be evaluated. Since many of the designs that successfully passed load tests used moments that were significantly lower than this sum, his paper evoked a spirited discussion that was over five times the length of the original paper. While some of the discussions applauded his analysis others, including Turner, questioned even the validity of applying statics to two-way slabs.

Those who were opposed to Nichols' analysis referred to the results of slab tests. Values of the total moment obtained from steel strain readings for six slabs representative of the many slab tests reported in terms of the total panel static moment, $M_o$, were

| | |
|---|---|
| Purdue test slab J | 0.59 $M_o$ |
| Purdue test slab S | 0.74 $M_o$ |
| Bell Street Warehouse | 0.40 $M_o$ |
| Western Newspaper Union | 0.72 $M_o$ |
| Sanitary Can Building | 0.30 $M_o$ |
| Shonk Building | 0.38 $M_o$ |

This apparent disagreement between the requirements of equilibrium and the results of tests was a dilemma that was perplexing to engineers and code writers.*

In 1917, the Joint Committee on Concrete and Reinforced Concrete included principles of design for flat slabs in their Final Report (5). Influenced by Nichols' logic but unable to ignore the results of the load tests, they compromised by adopting the form of Nichols' expression but arbitrarily reduced the coefficient and hence the magnitude of the total panel moment by recommending the expression

$$M_O = 0.107WL\,(1-\frac{2c}{3L})^2 \tag{2}$$

However, the approved 1920 ACI Building Code (6) defied statics even more by further reducing the coefficient to yield

$$M_O = 0.09WL\,(1-\frac{2c}{3L})^2 \tag{3}$$

Although this expression gives total panel moments that are only 72% of the total panel moment required to satisfy equilibrium, it remained in the ACI Building Codes essentially unchanged until 1971. The only modification was in the 1956 Code (7) where the total moment was multiplied by a factor F (where F = 1.15-c/L but not less than 1.0) to increase slightly the moments for slabs supported by small columns.

Initially, the only procedure specified for the design of flat slabs was known as the Empirical Method. In this method each panel was divided into column and middle strips and the total moment given by Eqn. 3 was proportioned to the different critical sections using specified percentages. It is interesting to note that the 1920 code specified the distribution for only 80% of $M_O$ leaving it up to the

* Later Westergaard and Slater (4) would show that the straight line method to compute moments from the measured steel strains greatly underestimated these moments as the effects of concrete tensile stiffening were not taken into account. Using statically determinate laboratory samples with similar reinforcement, they demonstrated that the sum of the actual moments corresponding to the measured steel strains were in close agreement with those predicted by Nichols' analysis. Unfortunately this information was not available to early code writers and, for many years, was ignored by others.

designer to distribute the remaining 20% "as required by the physical details and dimensions of the particular design employed". In the 1928 code and all following codes until 1971, designer discretion was removed and $M_O$ from Eqn. 3 was distributed as follows:

| | Negative Moment | Positive Moment |
|---|---|---|
| Column Strip | 46 | 22 |
| Middle strip | 16 | 16 |

## ORIGIN OF ELASTIC FRAME ANALOGIES

In 1929, a subcommittee of the reinforced concrete section of the Uniform Building Code, California edition, was established to investigate the idea of considering a flat slab and its supporting columns as a series of elastic frames. Although the report of this subcommittee was not published until 1938 by Dewell and Hammil (8), it did lead to the inclusion of an Elastic Frame Method as an alternative method for flat slab systems in the 1933 California edition of the Uniform Building Code. This method defined the frames as bounded by the centerlines of the panels as we do today. Columns were modeled as having hinges at the mid-height between the base of the capital and the top of the floor below. The column-slab joints were considered rigid.

Since the elastic frame analysis satisfied the equations of equilibrium, the design moments were considerably greater than those from the Empirical Method which accounted for only 72% of the static moment. This inconsistency was eliminated by arbitrarily reducing the negative moments obtained from the frame analysis by 40%. These positive and negative moments were distributed to column and middle strips in the same proportions specified for the Empirical Method.

An elastic frame method was first introduced into the ACI Code in 1941. The geometry for the equivalent frames followed closely that proposed by Dewell and Hammil except that the columns were assumed fixed at their far ends. The joints between columns and slabs were considered rigid. This rigidity was assumed to extend in the slabs a distance A, (where A was the distance from the center of the column, in the direction of the span, to the intersection of a 45 degree diagonal from the center of the column to the bottom of the flat slab or drop panel but not greater than one-eighth of the span) and in the column to the intersection of the sides of the column and the 45 degree line defining A. Negative moment was computed at a distance 0.073L +0.57A from the column center-line, this distance being selected to result in total panel moments for interior square panels with uniform loading that agreed closely with those that would be obtained with the Empirical Method. Thus the effect was essentially the same as merely reducing the negative moments by 40 % as specified by the UBC, California Edition, but was not as transparent to or as easy for the designer.

The elastic frame method was modified in the 1956 Code so that the rigid joint extended in slabs from the center of the column to the edge of the column or capital and in the column from the top of slab to the bottom of the capital and the distance from the column center at which negative moments were computed was simplified to the length A. With these changes the design moments remained essentially the same as before.

When the live load did not exceed three-quarters of the dead load, the design moments were assumed to occur with full live load on all spans. Otherwise, design moments were obtained with full live load on appropriate spans to give maximum values.

## THE ILLINOIS SLAB STUDY

The situation in the early 50's was that flat slabs, even when using an elastic analysis with full span geometry and design loads, resulted in design moments that were substantially less than those required by considerations of equilibrium. On the other hand, two-way slabs (slabs with beams) were designed using coefficients that were in part developed from elastic plate theory and so satisfied equilibrium. As it was generally recognized that there is no essential difference in the behavior between slabs with or without beams, the differences in design procedures and differences in factors of safety needed to be addressed.

To resolve this situation, the Reinforced Concrete Research Council initiated a comprehensive study of slab systems at the University of Illinois, Urbana. This study, begun in September, 1956, involved both laboratory testing and analytical studies. A paper by Sozen and Siess (9) outlines the scope of this study and why it was commissioned.

Five tests of nine panel 1/4 scale slabs with and without beams were tested to failure. These tests confirmed that there was no fundamental difference in their behavior and the existing design rules led to smaller factors of safety for flat slabs. Using the Newmark plate analog to generate difference equations for beams and columns, elastic solutions were obtained for similar slabs for purposes of comparing with the test slabs and for extending the parameters.

The results of this study led to a unified approach to the design of two-way slabs with and without beams in the 1971 ACI Code.

## 1971 ACI CODE PROCEDURES

Two procedures, the Direct Design Method (DDM) and the Equivalent Frame Method (EFM) were introduced to replace the five methods for slab design in the 1963 code. Both methods are essentially frame methods.

The DDM originally considered a design strip similar to the elastic frame but without the interior columns. The total factored panel moment was computed for each span as:

$$M_o = 0.125 w l_2 l_n^2 \qquad (4)$$

where w is the factored design load per unit area, $l_2$ is the span length perpendicular to the design strip and $l_n$ is the clear span length. For interior panels 65% of $M_0$ was assigned to the negative moment and 35% to the positive moment. For exterior spans, a modified frame analysis was performed by computing a factor, $\alpha_{ec}$, and distributing the moments in the exterior spans as functions of this factor as illustrated in Fig. 4. $\alpha_{ec}$ was defined as the ratio at a joint of the stiffness of the equivalent edge column (defined later for the EFM) to the stiffness of a slab-beam element based on gross concrete dimensions. Computing $\alpha_{ec}$ was extremely tedious and in the 1983 ACI Code this exterior column computation was replaced with a set of coefficients depending on the edge support. It now resembles the old Empirical Method except that the coefficient for $M_o$ is such that it gives a value of total panel moment that is much closer to a Nichols' type analysis.

The EFM replaced the former elastic frame analysis. As part of the Illinois study Corley, Sozen and Siess (10) compared the moments calculated from the elastic frame analogy defined in the 1956 ACI Code with known elastic solutions. They concluded that, in general, the elastic frame analysis gave values of the positive moment that were to low and values of the negative moment at the column centerline that were to high. Generally the design negative moments after reducing to the critical section were either too high or too low for interior columns depending of the dimensions of the panel and column and too high for exterior columns. To overcome these difficulties required proposing new stiffnesses for the members of the elastic frame. These new stiffnesses incorporated in the EFM were presented by Corley and Jirsa (11).

The concept is to introduce torsional members between the columns and slab-beam elements. The reduction in column stiffness is achieved by defining an equivalent column formed by the actual column and attached torsional members as shown in Fig. 5. Torsional members are assumed to have a constant cross sections throughout their lengths consisting of a portion of slab having a width equal to that of the column or capital plus that portion of the transverse beam above or below the slab, if any. A stiffness $K_t$ is computed from the expression

$$K_t = \Sigma \frac{9 E_c C}{l_2 (1 - \frac{c_2}{l_2})^3} \tag{5}$$

where the section parameter C is evaluated for the cross section by dividing it into separate rectangular parts and summing as follows

$$C = \Sigma (1 - 0.63 \frac{x}{y}) \frac{x^3 y}{3} \tag{6}$$

The stiffness of the equivalent column is defined as the sum of the flexibilities of the column and torsional member as

$$\frac{1}{K_{ec}} = \frac{1}{\Sigma K_c} + \frac{1}{K_t} \tag{7}$$

In computing the stiffness of the columns, $K_c$, the moment of inertia at any cross section outside the joint is based on the gross area of concrete taking into account any variation in section along the axis and at a joint is considered infinite from the top to the bottom of slab-beam. This is shown in Fig. 6.

Similarly, Fig. 7 shows the geometry for a typical slab-beam element where the moment of inertia is based on the concrete section outside the joint but from the center of the column to the face of the column or capital is assumed equal to the moment of inertia at the face of the column or capital divided by the quantity $(1-c_2/l_2)^2$.

Since the frame is defined using the centerline dimensions of the members, the negative moments at the column centerlines must be reduced to obtain the design moments at the critical section, defined generally at the face of the supports.

As with previous elastic frame analyses, when the specified live load was less than three-quarters of the specified dead load, design moments were obtained with full factored load on all spans. However, when the specified live load exceeds three-quarters of the specified dead load, the design moments are obtained with full factored dead load on all spans but only three-quarters of factored live load on appropriate spans to give maximum effects.

At the time the EFM was formulated, the only practical solution for elastic frame analysis was the moment distribution procedure, hence the need to determine fixed end moments, distribution factors and carry-over factors for the non-prismatic members resulting from the stiffness definitions. Although approximate values for these parameters for different geometric conditions have been tabulated to assist designers, for example Misic and Simmonds (12), the method is impractical for manual computation. However the concept has been incorporated successfully in computer programs written expressly for these definitions of stiffness and using a slope-deflection formulation.

In both the DDM and the EFM the last step is to distribute the design moments at the critical section across the width of the panel. In the 1971 Code, rules for this distribution to column and middle strips were given as part of the DDM. The definition of the column strip was defined with a width equal to half the smaller of $l_1$ or $l_2$ centered on the column line instead of half the frame width as was used in previous codes. The middle strip was the remainder of the slab width. The distribution rules were specified for exterior negative, interior negative and positive moment critical sections and were functions of the panel aspect ratios.

## LIMITS FOR APPLICATION OF ELASTIC FRAME ANALOGIES

Until the 1971 Code, the elastic frame method and all design specifications for flat slabs were explicitly limited to slabs with square or rectangular panels. All of the rules for assigning member stiffness and distributing design moments laterally across the slab both before and for the 1971 Code were developed by considering only square or nearly square panels.

Six limitations are listed before the DDM can be used. Three of them, namely, a minimum of three spans, limiting successive span lengths to one-third of the longer span and limiting the ratio of live to dead load are required for the DDM in order for the coefficients used to analyze the design strip to be valid. The remaining three limitations, namely, ratio of longer to shorter spans not greater than 2, column offset to maximum of 10% of span and limits to the effective beam stiffness ratio are required to ensure two-way behavior and the validity of the lateral distribution rules. As such these limitations must also apply for use of the EFM or any other elastic frame analogy.

While it may be argued that a frame can be defined for any irregular slab system and this frame analyzed for any arbitrary gravity loading including point loads, the use of elastic frame analyses as defined by the codes for other than rectangular panels should be viewed with caution. Certainly the lateral distribution of moments in irregular slabs may differ significantly from the rules given in the ACI code.

## FUTURE IMPROVEMENTS TO ELASTIC FRAME ANALOGIES

While there are many areas in which elastic frame analogies may be improved, only two, simplifying member stiffness for gravity loading and specifying member stiffness for lateral loading are mentioned here.

It is acknowledged that the attached torsional member concept developed for the EFM generally gives solutions that are closer to elastic solutions than previous frame analyses. However, the method is unnecessarily complex and the