Theory of Elastic Analysis—Illusion and Superstition

by R. W. Furlong

Synopsis: Building Codes have specified for the purpose of design, that *"Theory of Elastic Frames"* be used for analyses of indeterminate structures. Sophisticated computer software has been developed based on the condition of elastic response to loading. Designers rely on computed results from such analyses as if those results were perfect and reliable evaluations of structural behavior. The influence of assumptions regarding accepted simplifications, member stiffnesses, load definitions and frame connectivity is addressed. Reliability from elastic analyses is revealed as an illusion, and less sophisticated alternate analytic requirements are suggested.

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Richard W. Furlong, P.E., Ph.D.
Honorary Member ASCE
Emeritus Professor of Civil Engineering
The University of Texas at Austin

Introduction

"*All members of frames or continuous construction shall be designed for the maximum effects of factored loads as determined by the* **theory of elastic analysis***, except as modified according to 8.4.*" ACI 318-02, "Building Code Requirements for Structural Concrete[1], contains these words, and previous editions have contained the same mandate since Strength Design became part of the Code in 1963. Section 8.4 of ACI 318-02 permits negative moments determined from theory of elastic analysis to be modified by the designer within limits expressed in terms of anticipated flexural ductility. The Manual of Steel Construction – Load and Resistance Factor Design[2] – Section A5.1 permits the designer to use either elastic or plastic analysis to determine maximum effects of factored load. How did ACI318 reach its current position regarding analysis?

Basic civil engineering curricula in the United States before World War II offered analysis courses applied primarily to statically determinate systems. American college curricula for civil engineering began to include iterative moment distribution procedures[3,4] only after World War II. Applications of the theory of elastic analysis to indeterminate systems have been familiar to most American structural engineers for a mere 60 years, perhaps two generations of designers. Sixty years ago, the analytic effort with slide rules remained rather tedious. In order to reduce required analytic effort, Section 702 of ACI 318-47[5] and similar sections of subsequent issues permitted simplified substructures within monolithic concrete frames as acceptable for design.

The advent of digital computers after 1950's expanded the degree of complexity that designers could consider, but designer efforts were hardly less tedious for the first generation of engineers trained for indeterminate analysis. Early computer use was quite expensive, often with the computational help of rented time on large computers. During the 1980's, comprehensive digital computer programs began to appear on the desk of every designer, removing analytic tedium even for complex building structures. Today's third generation of designers can perform from their desktop computers elastic analyses with impressive calculation precision and elegant graphical displays of results. After the somewhat tedious process of defining loads and structures for the computer, elastic analysis has become relatively painless, and designers can enjoy the cozy illusion that computed numbers are precise with analytic results that are completely acceptable in accordance with current design rules.

How reliable or "accurate" are results based on elastic analysis? Is there any other basis for determining required strength? How precise are the safety indices that have been derived from an elastic analysis? Should more than one analytic model be required for analyzing response to the variety of loading conditions specified for a complete design of specific buildings or bridges?

The Structural Model for Elastic Analysis

Elastic analysis requires that force and deformation be proportional and that the structural system be completely resilient. After loading is removed, deformations from the loading disappear. Material and geometric properties of structural members are unchanged as loads are applied or removed, and superposition of various load cases always produces a corresponding superposition of deformations.

Analysis requires a definition for each of 6 parameters for every segment of every member, material stiffness E, Poisson's ratio ν, the geometric properties of area A, flexural shape I, shear shape A_v and torsion shape J . For purposes of this discussion let it be assumed that there are no loads that cause deformations involving significant twisting or unusually high shear. Without twisting, only 3 parameters, material stiffness E and geometric properties A and I need to be considered. Members themselves are generally represented as linear elements along which all force components act and all deformations occur. The linear elements are the geometric centroid axes of each line-element member.

Almost all members of building structures are perceived with a cross section that remains the same from one end to the other. Such members are classified as prismatic members, and the product EI and the product AE are constant from one end to the other. If the geometric properties A and I vary along the length of a member, the member is classified as non-prismatic. Analytic procedures are available to incorporate non-prismatic properties into an elastic analysis for a member. However, unless non-prismatic dimensions of members can be defined, reinforced concrete structures are considered to be composed of prismatic members.

What value should be used for E, for A and for I when reinforced concrete members are to be analyzed? Traditionally, any influence of steel reinforcement has been ignored during the frame analysis part of design, and values A and I have been based on the outline of the concrete cross section, commonly referred to as the "gross" section. Without any influence from steel reinforcement, a value for E based exclusively on concrete has been taken as the secant modulus $E_c = w_c^{1.5} 0.043\sqrt{f_c'}$ or $4700\sqrt{f_c'}$ (in MPa units for E and for f_c'). Thus, the products A_gE_c and I_gE_c have been used as stiffness values of prismatic members for frame analysis. A modest sophistication has been employed by some designers, who have used for beams a value I taken as only half that of the gross concrete outline. That sophistication was employed in recognition that beams were likely to be cracked by flexure under service load, whereas columns and walls were not likely to be cracked under service load conditions.

A further sophistication in ACI 318-47 permitted designers to use for design the forces at the faces of joints rather than the forces at the theoretical node at which line-element members actually intersect. Without additional advice regarding determination of "face-of-joint" forces, designers simply used the forces that would occur along prismatic members at the face of joint.

Prismatic members with or without modified proportions of stiffness or without simplified substructure models continue to be acceptable for the elastic analyses that is required by current ACI 318-02 design rules.

Monolithic Reinforced Concrete Structures - Only a Concept

Each of the simplifying assumptions accepted by ACI 318-02 results in some variance between analytic results and results obtained from more complex procedures that have not been simplified. Variances from specific influences will be examined.

Actual reinforced concrete members are not prismatic. From joint node to joint node, cross sections along any member must vary as the effective cross section between the joint nodal point and face of joint will be much larger than the effective section between faces of joints. "Exact" elastic models should employ non-prismatic members with greater stiffness through joint regions at the ends of members. Also, and of more significance, concrete cracks must form in flexural tension before reinforcement helps significantly to resist applied moments. Such flexural cracking not only reduces effective flexural stiffness at cracked sections, but cracking alters the position of the elastic centroid of the cracked section, such that the neutral axis along a linear member cannot be a straight line from end to end after flexural cracking.

The influence of longitudinal reinforcement should not be ignored. The amount and the position of longitudinal reinforcement are not known for any structural component at the time designers begin the process of structural analysis. Longitudinal reinforcement is assigned on the basis of results obtained from the requisite elastic analyses, such that the amount and the location of reinforcement is not the same at every position along the length of each member. Steel is 7 to 9 times stiffer than concrete, but the amount of steel in any section is only 1 to 2 percent of the area of concrete. Nevertheless, in sections before concrete cracks, longitudinal reinforcement increases flexural stiffness as much as 10% to 15% above the value for concrete alone. After concrete cracks in tension, steel reinforcement influences flexural stiffness significantly.

Beams that support monolithic slabs have a T-shaped section for which the slab helps resist compression in positive moment regions between supports, but for which slabs can be cracked in flexural tension near supports. Dramatic changes that can occur in response to cracking are illustrated by an example given in Fig.1. The figure displays graphs of bending stiffness EI together with graphs of bending moment along a continuous 3-equal-span beam under constant uniform loading. The (black) graphs connecting diamond-shaped data points represent the prismatic beam before any cracking. The moment diagram for the prismatic beam customarily would be used for designing the beam. Maximum moments at joints would be -160 kN-m, and positive moments as high as 125 kN-m occur in each end span.

The (pink) graphs that connect square data points represent the same beam and loading after a flexural crack forms at interior Support 2, reducing EI to half its value before cracking. Analytically, that one crack reduces the negative moment at the 2nd

support from -160 kN-m before cracking to -30 kN-m after cracking, while the moment at the 3rd support increases from -130 kN-m before cracking to -340 kN-m after cracking. The positive moments in the left span increase to 185 kN-m. Before this scenario would occur, the large moment over support 3 would cause a crack at the 3rd support. The (blue) graphs connecting x's represents the same beam and load with cracks over both interior supports. That graph is similar to the graph for response of the prismatic beam, except that the maximum negative moments at interior supports are smaller (-130 kN-m) and positive moments are larger (+130 kN-m) than were the values before any cracks occur. Positive moment in the center span increased from 40 kN-m before cracking to 80 kN-m after cracking at the supports..

The (yellow) graphs connecting triangular data points represent the beam with a 50% EI reduction due to a positive moment crack only in the left span. That graph indicates a modest reduction from +130 kN-m to +110 kN-m in the left span, a modest increase in negative moment at Support 2 from -160 kN-m to 170 kN-m. However, at Support 3 there is a large increase in negative moment from -130 kN-m to -280 kN-m. The large increase in moment in a support implies that the positive moment crack in an end span would be accompanied by cracking also at support regions.

Finally, a (brown) graph connecting a lined x (∗) represents the same beam and loading with positive moment cracks in both end spans and negative moment cracks over each interior support. That graph suggests that moments at supports increase from -160 kN-m before cracking to -205 kN-m after cracking while maximum positive moments in end spans decreased from 125 kN-m before cracking to 105 kN-m after cracking. There are no positive moments in the uncracked center span after cracking in end spans and at supports.

It is quite obvious that as reinforced concrete members become non-prismatic due to cracking, response to load differs significantly from response of the members before cracking. Flexural cracking reduces cross section effective flexural stiffness EI to values far less than the values for sections before cracking. An estimate of stiffness EI for sections after cracking can be derived for the section if specific reinforcement areas are known at the myriad of anticipated locations of flexural cracking. Even though elastic theory can be applied after a multitude of cracked sections might be defined, the process would be extremely more complex but no more certain of accuracy, than a study of prismatic members before cracking. Reinforced concrete flexural members are neither elastic nor prismatic under service loads, and their response diverges even further from that of prismatic models as loads approach strength limit values required for design. Is it logical to apply prismatic member elastic theory to unpredictably non-prismatic members in the design process? Is there an alternate to elastic analysis for determining reliable levels of strength in non prismatic reinforced concrete members?

How Precise Are Loads for Which Structures Are Designed?

Loads on buildings and other structures can be classified into three categories, 1) those which generate vertical forces due to gravitational action – gravity loads, 2) those which generate horizontal forces – lateral loads, and 3) those which create forces due to volume changes within the structural system – environmental loads.

Gravity loads include Dead Load, which is the self-weight of a structure plus the weight of permanent attachments to the structure. It is possible to estimate the magnitude and location of Dead Loads more accurately than the magnitude of any other loads. Even when magnitudes and locations of dead loads are known, estimates of their effect on indeterminate reinforced concrete structures are subject to the uncertainties described in the preceding section. All other gravity loads can be called Live Loads. The magnitudes and locations of Live Loads such as movable equipment and vehicular traffic can be estimated with precision. However, most buildings must be designed for estimated "occupancy" loads such as snow, rain, furniture, human assembly, floor cover and temporary storage. Since the magnitude and location of occupancy loads overall are impossible to define, estimates of such loading must be made. Building codes specify recommended magnitudes for many occupancy loads[6]. If the potential area to be loaded exceeds 20 or so sq meters, the intensity of live load is not likely to be maintained over larger areas. As loading area increases, occupancy Live Loads can be reduced from 100% to as little as 40% of the total possible magnitude if the area of occupancy loading is large enough. Forces from moving traffic and from vibrating equipment include some dynamic effects that may be estimated as a percentage of the static, non-vibratory weight. Totally apart from the vagaries of results from elastic analysis of indeterminate frames, no precise magnitude of any occupancy Live Load can be established.

Lateral loads include actions of soil, water and wind against structures as well as base displacements from earthquake motions. The magnitude and location of lateral loads are even less predictable than are those from gravity type loads. Lateral pressure from soil against walls is influenced by the nature of the soil, the stiffness of the wall, and by variables such as depth to water tables and superimposed overburden on backfill. Lateral pressures from wind, both positive and negative, are only as predictable as is a velocity and direction of wind, environmental turbulence, configuration and texture of a structure and the vertical elevation of exposed surfaces. Similarly, loading from earthquake motions is only as predictable as is the magnitude, direction, frequency and duration of ground motion. Since ground motion initiates movement of structures from inert positions to dynamically active conditions, the weight and location of inert masses together with the stiffness and strength of building components influence loading from seismic events.

Predicting the precise intensity of seismic, tornado and cyclone events is beyond current ability to predict the causative influences. Consequently, structural engineers must use "extreme case" models of such events, acknowledge that damage of building components will occur, and design structural elements with adequate toughness to sustain inelastic deformations without overall loss of capacity to support the structure.

Significant environmental loads on reinforced concrete structures can occur from shrinking or swelling of building components due to changes in temperature and humidity. Environmental loadings do not produce overall changes in the magnitude of either gravity or lateral loading, but they may increase tension or compression forces to values higher than values due to gravity or lateral loading. The consequences of such environmental loading can influence apparent serviceability, but the overall strength of structures is not changed if the structure is ductile enough to sustain inelastic deformation without significant loss of strength. Thus, structural engineers safely can employ such devices as expansion joints and separation from expansive soil in order to maintain acceptable serviceability and appearance of ductile structural members without superimposing design forces from environmental loading.

Strength, Safety and the Analytic Method

Columns and walls must be designed such that supported beams and slabs will sag visibly to warn of distress from overload before columns or walls fail. Structural safety then becomes a function of flexural behavior. Shear failure can occur before significant inelastic flexural deformation of sections takes place. Therefore, both the analytic basis for predicting shear failure and the coefficient that reflects reliability must be set to values less than those applied for flexural failure. It is mandatory that flexurally ductile failure precede any other type of failure.

Safety requires that there must be strength adequate to sustain required forces at every component of a structure. Tools of structural analysis are applied in order to establish values for required forces, and safety is insured after principles of materials and mechanics are used to insure that designed strength values exceed values of required force. Whereas the theory of elastic analysis requires that all members connected at a joint rotate through the same angle when forces change, the theory of plastic frames does not require rotational continuity at joints. Instead, the theory of plastic analysis recognizes that prior to flexural failure, structural components can endure inelastic rotational deformation without significant loss of strength when forces change. Of course, both elastic analysis and plastic analysis as well as any intermediate method must insure equilibrium of required force prior to failure.

Flexural strength at any reinforced concrete section is reached at the onset of compression spalling of concrete. Sections in which tension bars must stretch 2 to 3 times the yield strain of reinforcement before bars rupture will not experience compression spalling of concrete before section failure. Flexural strength in any beam can occur only after the formation of an unstable mechanism consisting of virtual hinge regions in 3 positions, one at each end of a beam span and one between the ends of the span. After 3 hinges form, no increase in load is necessary before unlimited tension strains occur and compression spalling ensues. For the special case of a simply supported span, the ends are always hinged. The third hinge position will occur wherever capacity to resist positive moment is reached between the ends of the span. Similar, if not identical, levels of safety can be obtained either from results of plastic analysis or from results from elastic analysis or from any intermediate limit load analysis that recognizes inelastic flexural rotations without significant loss of strength.

Structures Influenced by Lateral Displacement – Sway Frames

Structures for which lateral forces impose significant forces on columns must be considered as unbraced frames or frames that sway. Structures in which lateral bracing resists all lateral forces such that columns are not affected by lateral forces are considered to be braced or non-sway structures. Shear walls provide lateral bracing in non-sway reinforced concrete structures. Only gravity-type loads need to be considered for the design of beams and columns in non-sway structures. Under lateral loading, shear walls must be sufficiently stiff that lateral displacements between floors are so small that forces at beam and column joints are virtually unchanged from values under gravity loading alone. Inelastic displacements due to seismic loading of non-sway frames obviously compel beam and column members to help resist displacement, but the beam and column members can be designed on the premise that seismic-augmented loads act against a non-sway structure.

Structures without stiff lateral bracing rely on the flexural stiffness actions of beams and columns to resist lateral loads. When there is lateral displacement between two levels, structural mass on the displaced upper level increases overturning loading in proportion to the amount of lateral displacement. The increased overturning load in its turn again creates more overturning action, but in stable structures the process converges to a condition of static equilibrium. Iterative applications of elastic analysis can be employed to estimate the iterated magnitude of beam and column forces at joints. Analytic inconsistencies between actual values EI and values EI used for analysis as previously described for gravity loading apply also to behavior for lateral loading. The consequences of inaccuracy are similar to those for analysis of resistance to gravity type load; estimated forces at joints will differ from actual forces, but as inelastic deformation of ductile components occurs, failure cannot take place until the designed strength of components is reached.

Stiffness values used for analysis must be smaller than those actually anticipated prior to failure, in order that harmful effects from lateral displacements are not underestimated. After more than 5 decades of silence regarding acceptable stiffness assumptions for elastic analysis, ACI Committee 318 provided advice in ACI 318-95[9] with the stipulation that results from elastic analysis for lateral load cases are acceptable if stiffness values EI are not greater than $0.35I_gE_c$ for beams and cracked walls, nor $0.70I_gE_c$ for columns and uncracked walls. These stiffness values, intended for use to produce estimates of joint forces from elastic analyses of reinforced concrete structures subject to lateral loads, may be used also for analysis of response to gravity type loads. Second order displacements need not be considered in analyses of response to gravity load. It has been demonstrated that the actual stiffness values in beams change significantly in response to loads such that almost any flexural stiffness values including those specified in ACI 318-95, Section 10.11.1 can be useful for analysis of gravity load forces. Analytic results will be useful, but hardly any more “accurate” for design than any sets of forces in equilibrium at the required load condition.

Rational Estimates of Required Strength and Ductility

Every member of a structure must be strong enough to sustain the actions of loads that act on the structure. Columns and walls must be proportioned and reinforced such that thrust as well as flexural action and shear cannot cause visible cracks or rupture. Beams must be reinforced such that tension forces from shears, joint moments and span moments never cause visible cracks or rupture. Estimates of required strength can be obtained with the theory of elastic analysis, but such estimates also can be obtained with the theory of plastic analysis.[7] The theory of elastic analysis requires that deformations be compatible at joints, but that stipulation is not required with plastic analysis. Reinforced concrete beams easily can be reinforced for plastic moment capacities that need not be the same at every point along a beam span. However, both positive and negative values of plastic moment are the same when plastic analysis is applied with structural steel rolled shape members.

The degree of flexural ductility necessary for hinge mechanisms to develop under gravity loading is greater when all plastic moments on a beam are the same, than the flexural ductility necessary if plastic moments can be proportioned with negative plastic moments larger than positive plastic moments. Concrete cross sections may have the same shape throughout continuous beams, but the amount of flexural tension reinforcement determines plastic moment capacity. Thus, it is customary and simple to design for different plastic moment capacities for positive and for negative moments of reinforced concrete beams. Under the action of gravity loading, the basic moment value for spans designed by plastic analysis is the magnitude of maximum moment in the span under the action of required transverse loading. That quantity will be designated as M_{su}, the maximum moment created by required loads on the span if it were simply supported.

It has been demonstrated that reinforced concrete beams can be designed for gravity loading with moment capacities simply assigned on the basis of desired limit strength (plastic moments).[8] In order to insure adequate strength and ductility until plastic hinge mechanisms develop, some restrictions must be specified if limit design with plastic analysis is permitted for constant depth reinforced concrete beams that support only gravity loads:

a) Reinforcement shall be continuous across interior supports extending at least the tension bar development length beyond each face of the support.

b) Plastic moment values:

Negative moment capacity at ends of spans shall be no less than $0.6M_{su}$.
Positive moment capacity within spans shall be no less than $0.4M_{su}$.

c) Shear capacity values:

Interior spans shall be adequate to sustain required loading on spans with no rotational restraint at either end.
The exterior end of exterior spans shall be adequate to sustain required loading on exterior spans with no rotational restraint at either end.
The interior end of exterior spans shall be adequate to sustain 115% of required loading the spans with no rotational restraint at either end.

Reinforced concrete columns can be designed for gravity loading with axial forces above each level based on contributing areas and with required end moments not less than $0.3M_{su}$ for either span supported by the column at the same level.

The plastic analysis procedure is not well-suited to applications with combined loading on multi-story frames that can sway. Not only must the most severe load combinations be considered, but the sequence of loading also must be considered for plastic analysis. Even with its known inaccuracies, an elastic analysis procedure may offer the least complicated analytic procedure for estimating forces required to design members in frames that sway.

The theory of elastic analysis can provide reliable estimates of strength required at all critical points of reinforced concrete structures. Numerous reasons have been given to explain why such estimates differ from values that would occur if required loading actually were applied to members of the structure. Discrepancies between estimated and "actual" design forces may exceed 25% of the estimated values, but flexurally ductile concrete members will deform inelastically to accommodate required loading without rupture. Discrepancies between estimated and "actual" service load forces, even with environmental loading, generally will not be large enough to reveal undesirable cracking. Also the theory of elastic analysis offers the least complex technique to be used for the various load combinations required for designing members that resist combined load cases on frames that sway.

Conclusion and Recommendation

The magnitude of internal forces necessary for designing reinforced concrete structures can be derived either from applications of theory of elastic frames or from theory of plastic analysis or from any limit strength condition between elastic response and fully plastic response. The accuracy, here defined as the ratio between estimated internal force and actual internal force, can vary from 2 to ½ when the theory of elastic frames is employed for analysis. A similar accuracy can be obtained from analyses based on the assignment of limit forces or from a plastic analysis of response to required loads.

An elastic analysis for estimated forces will produce values that differ significantly from actual forces that can occur within flexurally cracked members. Although precision from such analysis is an illusion, the procedure identifies for all possible load conditions the limit strength values at all critical locations. If identified strength is provided at each critical section, and general detailing requirements insure ductility, failure cannot occur before any one of numerous possible load cases acts against a member.

The magnitude of internal forces necessary for designing reinforced concrete structures also can be derived from applications of plastic analysis. A plastic analysis for estimated forces necessary for designing <u>laterally braced</u> (non-sway) reinforced concrete structures may be less complex and easier to apply than elastic analysis. General detailing requirements insure ductility adequate for reinforced concrete members to