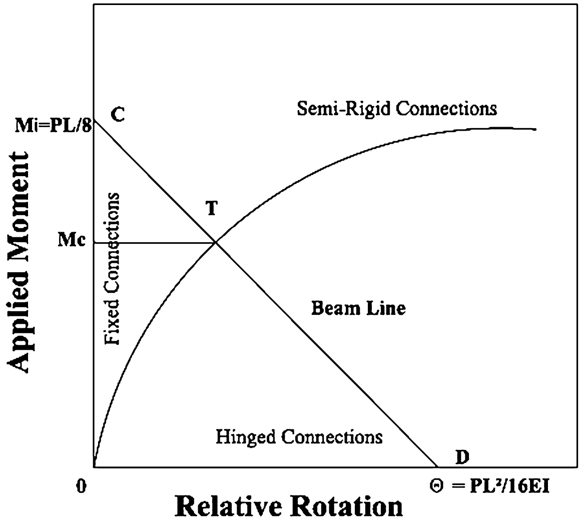


Figure 8-4. Beam-line method for semi-rigid connections analysis.

rotation can be calculated using virtual work or other techniques (in this example, $\Theta = PL^2/16EI$ as shown in Fig. 8-4). These two conditions produce two points on the linear M/θ curve of the beam (at the x and y axes). The line connecting these two points (C and D) is called the beam-line, as shown in Fig. 8-4. The intersection of the beam line CD and the connection M/Θ curve gives the moment and rotation that are actually present in the member. It should be noted that, for this procedure, the end deformation of the column was ignored based upon the assumption that a relatively small deformation will occur at the column end. The column-end deformation is prevented from rotation by the beam moment acting on the opposite side. For design, the actual value for the negative beam end moment, M_e, can be determined from the graph. This value falls between zero and $wL^2/12$ for a uniformly loaded beam. See Mosallam (1994) for discussion on the end moment for semi-rigid connections. Mottram and Zheng (1996) have adopted this beam-line technique to describe PFRP interior beam-to-column connections.

8.3.4. Closed-Form Expressions for Beams with Semi-Rigid End Connections

Simple expressions for deflection and end rotations of composite beams with semi-rigid behavior, which accounts for shear deformation, was proposed by Turvey (1998). In developing this closed-form expression, the moment–rotation behavior was assumed to be linear. This closed-form expression is given by:

$$\delta = \frac{QL^3}{k_1 E_{11} I_{11}} \left(\frac{1 + 48\alpha + k_2\beta + 96\alpha\gamma}{1 + 2\beta} \right) \tag{8-2a}$$

or

$$\delta = \phi \left(\frac{1 + \varpi}{1 + 2\beta} \right) \tag{8-2b}$$

where

$$\phi = \frac{QL^3}{k_1 E_{11} I_{11}} \tag{8-3a}$$

$$\varpi = 48\alpha + k_2\beta + 96\alpha\beta \tag{8-3b}$$

and

δ = mid-span deflection
L = span
Q = total applied load
E_{11} = longitudinal modulus of elasticity of the beam
I_{11} = moment of inertia (major axis)
k_1 and k_2 = constants that depend on the load distribution (refer to Table 8-1)
β = dimensionless connection flexibility parameter expressed as:

$$\beta = \frac{E_{11} I_{11}}{K_i L} \tag{8-4}$$

where

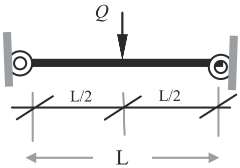
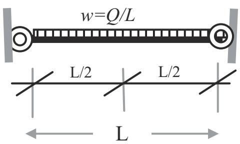
K_i = initial linear rotational stiffness of the connection which is determined from the M/θ experimental curve as described earlier
α = dimensionless shear flexibility parameter expressed as:

$$\alpha = \frac{E_{11} I_{11}}{k^V G_{21} A L^2} \tag{8-5}$$

where

k^V = modified shear coefficient. Different expressions for calculating this coefficient can be found in Bank and Bednarczyk (1988) and Mosallam and Chambers (1995).
G_{21} = shear modulus
A = cross-sectional area.

Table 8-1. Values of Coefficients k_1, k_2, k_3, and k_4 for Semi-Rigid PFRP Beams with Semi-Rigid End Connections

Beam Loading Distribution on Beams with Semi-Rigid Ends		k_1	k_2	k_3	k_4
	Point load at mid-span	192	8	8	4
	Uniform load over entire span	384	10	12	5

If the shear deformation effect is ignored (i.e., $\alpha = 0$), Eq. 8-2 will be reduced to the following simpler form:

$$\delta = \frac{QL^3}{k_1 E_{11} I_{11}} \left(\frac{1 + k_2 \beta}{1 + 2\beta} \right) \tag{8-6}$$

Equation 8-6 will reduce to the commonly known mid-span deflection expressions for simply supported beams by setting $K_i = 0$, and consequently, $\beta = \infty$:

$$\delta = \frac{QL^3 k_2}{2k_1 E_{11} I_{11}} \tag{8-7}$$

Similarly, the mid-span deflection expression for fixed end beams (without a shear deflection component) can be obtained from Eq. 8-6 by setting $K_i = \infty$, and consequently, $\beta = 0$:

$$\delta = \frac{QL^3}{k_1 E_{11} I_{11}} \tag{8-8}$$

Turvey (1997) also developed the following expression for calculating the rotation of the semi-rigid ends of a composite beam:

$$\theta = \frac{QL^2}{k_3 E_{11} I_{11}} \left(\frac{\beta}{1 + 2\beta} \right) \tag{8-9}$$

It should be noted that, as expected, the shear deformation has no effect on Eq. 8-9. Also, for a beam with fixed ends (i.e., $K_i = \infty$, and consequently $\beta = 0$), Eq. 8-8 reduces to zero, and by setting $K_i = 0$, and consequently, $\beta = \infty$, the following expression is obtained from Eq. 8-9 describing the end rotations of a simply supported composite beam:

$$\theta = \frac{QL^2}{2k_3 E_{11} I_{11}} \tag{8-10}$$

8.3.4.1 Performance Indices. To appreciate the gain of including the partial fixity (semi-rigidity) of commonly used connection details of PFRP frame connections, Turvey (1997) proposed expressions for what are called "performance indices" that relate the mid-span deflection, associated load, and end rotations of composite beams with semi-rigid ends and semi-rigid beams to identical composite beams with simply supported end conditions (which is commonly used today in sizing PFRP frame structures members). These coefficients are similar to the λ-coefficients introduced initially by Mosallam and Chambers (1995) to relate the long-term total deflection to short-term instantaneous deflection of PFRP beams.

8.3.4.1.1 Deflection Reduction Index (λ_δ). The deflection reduction index is the ratio between the mid-span deflections of a beam with a specific semi-rigid rotational stiffness and a simply supported beam having identical properties, dimensions, and subjected to the same total load (Q). This expression is obtained by dividing Eq. 8-2 by the same equation after setting $\beta = \infty$ (a simply supported case). Introducing a new deflection factor k_4 (refer to Table 8-1) and rearranging, we get:

$$\lambda_\delta = \left(\frac{1 + 48\alpha + k_2\beta + 96\alpha\beta}{k_4 + 48\alpha + k_2\beta + 96\alpha\beta} \right) \tag{8-11}$$

or

$$\lambda_\delta = \frac{1 + \varpi}{k_4 + \varpi} \tag{8-12}$$

If the shear deformation component is neglected (i.e., $\alpha = 0$), Eq. 8-11 will reduce to:

$$\lambda_\delta = \left(\frac{1 + k_2\beta}{k_4 + k_2\beta} \right) \tag{8-13}$$

As expected, λ_δ equals unity for the case of a simply supported beam (i.e., $\beta = \infty$). On the other hand, if the beam's ends are fixed, Eq. 8-11 reduces to:

$$\lambda_\delta = \left(\frac{1}{k_4}\right) \tag{8-14}$$

Equation 8-14 results in the known ratio between fixed-end and simply supported mid-span deflections of identical beams subjected to identical loads.

8.3.4.1.2 Load Enhancement Index (λ_Q). The load enhancement index is the ratio between the load capacity of a beam with a specific semi-rigid rotational stiffness and a simply supported beam having identical properties, dimensions, and subjected to the same deflection limit (e.g., $\delta_{max} = L/360$). This can be obtained by rearranging the two forms of Eq. 8-2. It is obvious that this enhancement index is simply the inverse of the deflection reduction index (λ_δ), in general, regardless of the type of end conditions or the inclusion of the shear deformation effects, that is,

$$\lambda_Q = \left(\frac{1}{\lambda_\delta}\right) \tag{8-15}$$

As expected, λ_Q equals unity for the case of a simply supported beam (i.e., $\beta = \infty$). On the other hand, if the beam's ends are fixed, Eq. 8-11 reduces to:

$$\lambda_Q = k_4 \tag{8-16}$$

Equation 8-16 results in the known ratio between fixed-end and simply supported load capacities of identical beams subjected to identical span/deflection limits; that is, a fixed beam loading capacity is k_4^{th} the capacity of an identical beam with same span-to-deflection limit.

8.3.4.1.3 Span Enhancement Index (λ_L). The span enhancement index for a composite beam with a prescribed load and a mid-span deflection limit is the ratio between the allowable span of a beam with a specific semi-rigid rotational stiffness and the allowable span of a simply supported beam having identical properties, dimensions, and subjected to the same loads and mid-span deflection limit. This can be determined as the positive root of the following cubic equation:

$$\lambda_L^3 + k_2\lambda_L^2\beta - k_4\lambda_L - k_2\beta = 0 \tag{8-17}$$

For a composite beam with a fixed end, i.e. $K_i = \infty$, and consequently $\beta = 0$, Eq. 8-17 reduces to the following simpler form:

$$\lambda_L^3 + k_4\lambda_L = 0 \tag{8-18}$$

The roots of Eq. 8-18 are: $\lambda_L = -\sqrt{k_4}$, $\lambda_L = +\sqrt{k_4}$, and $\lambda_L = 0$. Thus, the span enhancement index is $\lambda_L = +\sqrt{k_4}$ (the only positive root). Similarly, for a case of a simply supported beam, i.e. $K_i = 0$, and consequently, $\beta = \infty$, Eq. 8-17 will be simplified to the following form:

$$\lambda_L^2 - 1 = 0 \tag{8-19}$$

Solving Eq. 8-19 yields the following two roots: $\lambda_L = \pm 1$. Using the positive root, the span enhancement index, as expected, is equal to unity.

8.3.4.1.4 Rotational Capacity (θ_c). It is advantageous to express the rotational capacity of a composite beam in terms of the serviceability limit on mid-span deflection, that is, as a function of the prescribed deflection-to-span ratio, κ_c (Turvey 1997). An expression for the rotational capacity, θ_c, is obtained by combining Eqs. 8-2 and 8-8, and replacing $\frac{\delta}{L}$ by the deflection-to-span ratio κ_c. Thus:

$$\theta_c = \frac{k_1}{k_3}\left(\frac{\kappa_c\beta}{1+\varpi}\right) \tag{8-20}$$

If the shear deformation effect is ignored, i.e., $\alpha = 0$, then Eq. 8-20 reduces to:

$$\theta_c = \frac{k_1}{k_3}\left(\frac{\kappa_c\beta}{1+k_2\beta}\right) \tag{8-21}$$

To demonstrate the effectiveness of the aforementioned closed-form equations for determining both the mid-span deflection and the end rotations of a composite beam with semi-rigid end connections, two numerical examples are presented.

Example 8-1: Calculate the mid-span deflection and the end rotations of an 8 in. × 8 in. × 3/8 in. (203 mm × 203 mm × 9.5 mm) PFRP E-glass/vinylester H-beam (Pultex 1625) with semi-rigid end connection details (refer to Fig. 8-5 here and Fig. 7-20D in Chapter 7 for end connection details). The total factored load is 2,500 lb (11,120 N) applied at the mid-span. The total span of the PFRP beam is 9 ft (2.74 m). The initial linear

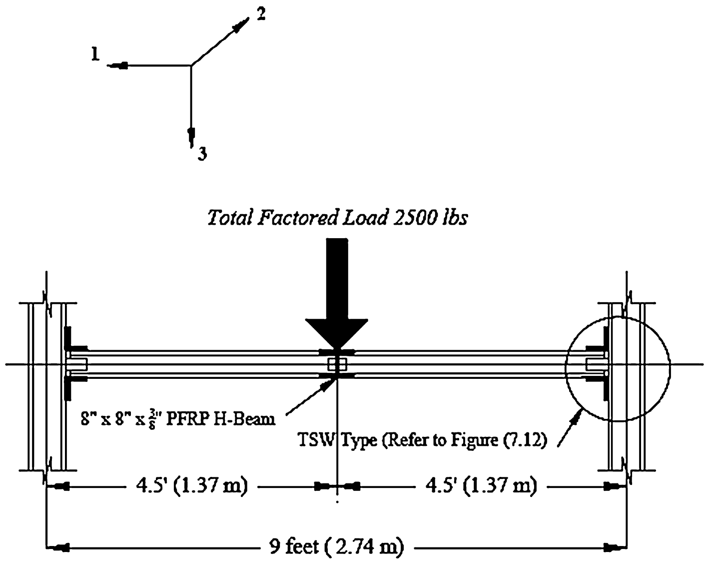


Figure 8-5. Details of the semi-rigid connected PFRP beam of Example 8-1.

rotational stiffnesses of the end connections were determined experimentally [refer to connection detail TSW in Fig. 7-21 and Table 7-3 to be 7,000 kip-in./rad (790.3 kN-m/rad)]. The following are the PFRP beam mechanical properties that were measured experimentally (Mosallam 1990): $E_{11} = 2.35 \times 10^6$ psi (16.20 GPa), $E_{22} = 1.00 \times 10^6$ psi (6.90 GPa), and $G_{21} = 0.54 \times 10^6$ psi (3.72 GPa).

SOLUTION

i) **Section Properties:** Using the *Creative Pultrusions Design Guide* tables (Creative Pultrusions, Inc. 2003), the major moment of inertia and the cross-sectional area of the beam pultruded profiles are $I_{11} = 99.18$ in.^{4} (4,127.8 cm^4); and $A = 8.73$ in.^{2} (56.31 cm^2), respectively.

ii) **Calculate the Modified Shear Coefficent (k^v):** Due to the anisotropic nature of PFRP composites, mechanical properties are directionally-dependent. As a consequance, the ratio of the in-plane longitudinal modulus, E_{11}, to the in-plane shear modulus, G_{21}, for the pultruded profiles is higher than that of isotropic materials. In our case, $E_{11}/G_{21} = 4.35$ as compared to a ratio of 2.60 for isotropic materials. For this reason, it is recommended to consider the shear deformation component when calculating the total mid-span deflection (Mosallam 1990). The total deflection at any point along the beam span is calculated using the following equation (Mosallam and Bank 1992):

$$\delta_{total} = \frac{f_F}{E_{11}I_{11}} + \frac{f_V}{k^v AG_{21}} \tag{8-22}$$

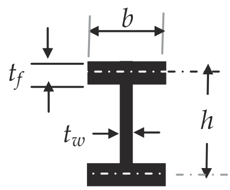
where

δ_{total} = total deflection due to bending moments and shear forces

f_F and f_V = functions that depend upon the loading and the boundary conditions (refer to Table 8-2). Note that subscript F refers to the flexural term, and subscript V refers to the shear term

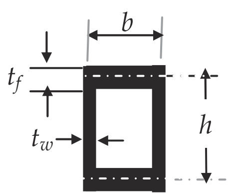
k^v = the modified shear correction factor (Bank and Bednarczyk 1988).

For H-beams (open-web profiles):



$$k^v = \frac{20(\zeta+3m)^2}{\left[\frac{E_f}{G_f}(60m^2n^2+60\zeta mn^2)+\frac{E_f}{G_w}(180m^3+300\zeta m^2+144\zeta^2 m+24\zeta^3) + \nu_f(60m^2n^2+40\zeta mn^2)+\nu_w(30m^2+6\zeta m-4\zeta^2)\right]} \tag{8-23}$$

For box-beams (closed-web profiles):



$$k^v = \frac{20(\zeta+3m)^2}{\left[\frac{E_f}{G_f}(60m_b{}^2n^2+60\zeta m_b n^2)+\frac{E_f}{G_w}(180m_b{}^3+300\zeta m^2+144\zeta^2 m_b+24\zeta^3) + \nu_f(-30m_b{}^2n^2-50\zeta m_b n^2)+\nu_w(30m_b{}^2+6\zeta m_b-4\zeta^2)\right]} \tag{8-24a}$$

where

$$n = \frac{b}{h}$$

$$m = \frac{2bt_f}{ht_w}$$

$$m_b = \frac{bt_w}{ht_f}$$

t_f = flange thickness
t_w = web thickness
b = flange width
h = distance between the centerlines of the flanges

$$\zeta = \frac{E_w}{E_f}$$

E_w, G_w = longitudinal modulus of elasticity and shear modulus of the web
E_f, G_f = longitudinal modulus of elasticity and shear modulus of the flanges
v_f and v_w = Poisson's ratios for the flanges and the web, respectively.

It should be noted that, for prelimanary analysis, the shear correction factor can be taken as:

$$k^v \cong \frac{A_{web}}{A_{Gross}} \tag{8-24b}$$

where A_{web} = the area of the web(s), and A_{Gross} is the gross sectional area = $A_{web(s)} + A_{\ flanges}$.

Using Eq. 8-23, we have:

$$k^v = 0.29$$

Note: The approximate value of this coefficent according to Eq. 8-24b is:

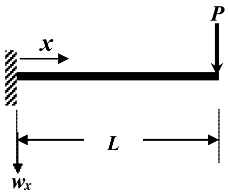
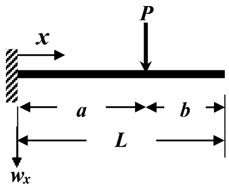
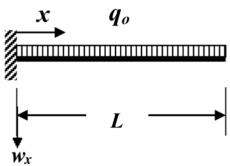
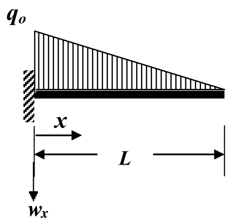
$$k^v \cong \frac{A_{web}}{A_{Gross}} \cong \frac{7.25 \text{ in.} \times 0.375 \text{ in.}}{8.73^2 \text{ in.}^2} = 0.31$$

iii) Calculate the shear flexability ratio:

Using Eq. 8-5, we get:

For $k^v = 1$:

Table 8-2. Flexure and Shear Deformation Functions

Load Case	Boundary & Loading Conditions	$f_1(x)$	$f_2(x)$
Load Case 1		$\frac{P}{6}\left(-x^3+3Lx^2\right)$	Px
Load Case 2		$\frac{Px^2}{6}(3a-x);\ 0 \le x \le a$ $\frac{Pa^2}{6}(3x-a);\ a \le x \le L$	$Px;\ 0 \le x \le a$ $Pa;\ a \le x \le L$
Load Case 3		$\frac{q_o}{24\left(x^4-4Lx^3+6L^2x^2\right)}$	$\frac{q_o}{2\left(-x^2+2Lx\right)}$
Load Case 4		$\frac{q_o x^2}{120L\left(10L^3-10L^2x+5Lx^2-x^3\right)}$	$\frac{q_o}{6L\left(3L^2x-3Lx^2+x^3\right)}$