Zdeněk Bittnar Jiří Šejnoha

NUMERICAL METHODS IN STRUCTURAL MECHANICS

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ABSTRACT:

This book provides a clear understanding of the nature and theoretical basis of the most widely used numerical methods—the finite element method (FEM) and the boundary element method (BEM)—while at the same time presenting the most promising directions for future developments. Attention is paid mainly to those methods that have proven to be the most reliable and efficient, as well as those methods currently under rapid development. Examples were selected either to illustrate various computational algorithms and compare their accuracy and efficacy or to elucidate the mechanical processes under investigation, while traditional examples that are already covered by standard textbooks have been deliberately omitted. Emphasis is placed on the understanding of basic principles, rather than on the details of individual numerical algorithms. The book covers all topics essential for students of elementary and intermediate courses on numerical methods in solid mechanics, and it also serves as a useful reference for researchers and other professionals. This book was recently translated from the highly regarded, original Czech edition.

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Preface

Fast development of numerical methods in mechanics has been attracting an increasing number of students, researchers and design specialists from all branches of engineering. A number of distinguished authors published books dealing with numerical methods in mechanics during the past decade. Contributions of K. J. Bathe; J. H. Argyris and H. P. Mlejnek; M. A. Crisfield; T. J. R. Hughes; E. Hinton and D. R. J. Owen; J. T. Oden; and O. C. Zienkiewicz and R. L. Taylor are among the most widely respected ones.

The aim of the present book is to help the reader in understanding the nature and the theoretical basis of the most widely used numerical methods—the finite element method (FEM) and the boundary element method (BEM)—and, at the same time, to sketch the most promising directions of their future development. Of course, it is hardly possible to cover all of the topics in this broad area in full detail. Attention is paid mainly to the most efficient and reliable methods which have become widely popular, and to methods which are currently under fast development. This is also reflected by the selection of examples, which either illustrate various computational algorithms and compare their accuracy and efficiency, or elucidate the mechanical processes under investigation. Traditional examples covered by standard textbooks (related, e.g., to the linear theory of plates and shells, or linear stability and vibration analysis) have been deliberately omitted.

In the authors' opinion, the book covers all the topics essential for students of elementary and intermediate courses on numerical methods in solid mechanics, and, in addition, it gives an overview of the most vital areas of current research. Problems not directly related to solid mechanics (e.g., problems of electric and magnetic potential, linear fluid mechanics, high speed gas flow, coupled problems, shallow water equations and wave propagation) as well as hints on programming have been omitted. On the other hand, we offer a detailed presentation of the fundamental equations in solid mechanics with emphasis on constitutive equations including quasibrittle materials, inspired by the voluminous textbook *Stability of Structures* by Z. P. Bažant and L. Cedolin. This relatively new area is likely to affect design methods in the near future and it should be brought to the attention of engineering students interested in numerical methods.

The present book also thoroughly discusses models of beams and plates continuously supported by an elastic foundation, which have many applications in geotechnical engineering, and probabilistic methods applicable, e.g., to slope stability analysis. In addition to FEM, the book explains the fundamentals of BEM (including its symmetric version and combination with FEM) as an alternate numerical method with important advantages over FEM in certain situations.

Emphasis is placed on the understanding of basic principles rather than on the details of individual numerical algorithms. The authors' intention was to educate the reader and help him or her to develop analytical skills necessary for conceptual thinking. We hope that this aspect will make our book a useful complement to the existing publications, most of which deal mainly with specific applications of FEM in mechanics.

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Introduction

The material in this book is divided into two parts.

Part I can be studied by readers who have acquired basic knowledge in elementary courses such as Strength of Materials, or Structural Analysis. It consists of five chapters.

Chapter 1 is a review of basic notions, relations and principles of solid mechanics. It should not only facilitate further reading but also make the reader aware of new trends in nonlinear material modeling. Problems related to damage localization, size effect, etc., are so important that, despite the limited scope of this book, the authors at least briefly explain their essence and give the appropriate references.

Chapter 2 is devoted to skeletal structures (trusses, frames and grillages) with special attention to soil-structure interaction. It presents a consistent derivation of the stiffness matrix of an elastic foundation based on the Winkler-Pasternak model, which is later used in linear stability and vibration analysis. Attention is also paid to curved beam elements based on the principle of decomposition of membrane and bending effects. The chapter is concluded by remarks on static condensation and on coordinate transformation.

Chapter 3 represents the core of the part devoted to linear problems. After an initial introduction to isoparametric elements, a thin-walled beam element based on the Umanski-Mindlin-Reissner hypothesis is derived. The next section presents elements for plane problems (plane stress or plane strain analysis) with several useful modifications, which can be exploited when analyzing deep beams, when combining in-plane loaded plates with frames, and when constructing efficient shell elements. Elements for plate bending (optionally supported by an elastic Winkler-Pasternak foundation) are derived from Kirchhoff theory, and from Mindlin-Reissner theory. The curved beam element based on the principle of decomposition from Chapter 2 is generalized to a shell element. The last portion of the chapter deals with special elements for subgrade modeling in soil-structure interaction analysis. Chapter 4 generalizes plane elements to three-dimensional solid elements.

Chapter 5 is devoted to linear stability and vibration analysis. Aside from standard methods (Rayleigh-Ritz method, inverse iteration, Jacobi method, subspace iteration method), Lanczos method is thoroughly discussed. Forced vibrations are analyzed by eigenmode decomposition (with special emphasis on alternate models for damping), and by direct integration (central difference scheme, Newmark method and Wilson method). The latter approach is applicable to linear as well as nonlinear equations of motion. Two methods of finding a periodic response to a harmonic excitation (the solution in complex numbers and the eigenmode decomposition) are then explained, and their applicability to models with proportional and nonproportional damping is discussed.

Part II has been designed for readers who are already familiar with methods of linear finite element analysis. It consists of six chapters covering three main subjects: special linear problems solved by FEM (Chapters 6 and 7) and BEM (Chapter 8), nonlinear problems (Chapter 9), and some modern topics (adaptivity in Chapter 10 and probabilistic approach in Chapter 11).

Chapter 6 presents semianalytical solutions based on Fourier expansion in one direction and finite element discretization in the other (orthogonal) direction. Methods of this kind (finite strip methods) are applicable, e.g., to curved box girders.

Chapter 7 deals with other special applications of FEM. Analysis of warping torsion is followed by diffusion problems (heat conduction and moisture transport). A similar numerical approach is applied in the analysis of deformation of soils and other porous materials. Some problems of linear elastic fracture mechanics are also included, and they are supplemented by comments on nonlinear fracture mechanics.

Chapter 8 explains basic ideas of the boundary element method and its modifications. It tackles both static and dynamic problems with special emphasis on recent developments leading to a symmetric version of BEM, which has important advantages when combining BEM with FEM.

Chapter 9 shifts the focus to nonlinear problems. It addresses both geometric and material nonlinearities. Geometrically nonlinear effects are demonstrated by an elementary example of a truss element. The basic notions are then generalized for a continuum, and the Total Lagrangian and Updated Lagrangian formulations using the incremental form of the principle of virtual displacements are explained. The discretization procedure is then generalized to isoparametric elements of an arbitrary shape and supplemented by comments on discretization of a degenerate continuum (arches and shells). Special attention is paid to modern solution methods for sets of nonlinear equations. Besides being very efficient, these methods are applicable even to problems for which the standard Newton-Raphson technique with load control fails (snap-through, snap-back). This section also includes basic facts on stability analysis of individual branches of the equilibrium diagram. BEM has some advantages when applied to problems with material nonlinearity. Dual formulations based on initial strain and initial stress concepts are presented and discussed.

Chapter 10 is devoted to the currently very popular area of adaptive meshes, especially to hierarchical elements and the p-version of FEM. The mathematical theory of FEM has provided reliable error estimators. Based on an error estimate, the mesh can be modified so that the error is approximately uniform. Applications of artificial intelligence to adaptive remeshing are briefly discussed and illustrated by an example.

Chapter 11 gives an overview of probabilistic methods used in combination with FEM or BEM, which include statistical methods (Monte Carlo simulation, stratification LHS method) and nonstatistical methods (probabilistic FEM).

The book is appropriate for undergraduate students on senior level (Volume I) and for graduate students (both parts). In the authors' opinion, it provides material for up to four courses—fundamentals of linear FEM, dynamic analysis, nonlinear problems and special topics.

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