This formulation is based on the theory of random processes. It includes the correlation of the individual eigenmodes, and it can be used even for structures with closely clustered frequencies. The research report of the Earthquake Engineering Research Center of the University of California #08 of 1991 presents further generalizations that include also the influence of local geological conditions in addition to the correlation of the individual modes.

5.4 Response to harmonic excitation

The subject of this section is the solution of the basic equation of motion of elastic solids discretized by the FEM for harmonic excitation. This loading is one of the most often used idealizations in structural dynamics, especially for structures loaded by rotating machinery. The solution can be obtained

- by direct solution in complex eigenvalues (or by expressing displacements in terms of amplitude and phase shift), or
- by mode decomposition.

Both approaches exhibit their pros and cons.

The first approach is more general with respect to the characteristics of damping, as it allows for nonproportional damping (it can be even said that the form of damping is immaterial). However, due to uncertainties in input data the solution cannot be carried out only for a single frequency. Rather, it is necessary to carry out the analysis for a frequency band in order to construct the resonance curve, and so the computation can be expensive. (Each point of the curve corresponds to one computation run.)

The efficiency of the mode decomposition is strongly dependent on the characteristics of damping. If the damping can be assumed to be proportional, the method is very efficient; otherwise the economy deteriorates.

Before describing the individual methods, the loads need to be classified. The total load is harmonic if all of the acting forces are harmonic with the same frequency (the *forcing frequency*). This can be achieved by decomposing each load component into two loads phase-shifted by $\pi/2$. The simplest form reads

$$\boldsymbol{R}(t) = \boldsymbol{R}_1 \cos\omega t + \boldsymbol{R}_2 \sin\omega t. \tag{5.165}$$

In the case of a steady harmonic vibration the usual parlance uses amplitude and phase shift to describe the load. In that case the nth component of the vector $\boldsymbol{R}(t)$ can be expressed by using amplitude a_n and the phase shift φ_n in the following manner (see Fig. 5.6):

$$R_n(t) = a_n \sin(\omega t + \varphi_n). \tag{5.166}$$

The relation between a_n, φ_n and the components of the vectors $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ is given by

$$a_n = \sqrt{R_{1n}^2 + R_{2n}^2}, \qquad \varphi_n = \arctan\frac{R_{1n}}{R_{2n}}. \tag{5.167}$$

As the solution is sought for a steady state, in which the eigenvibration has already vanished from the response, the time-dependency of $\boldsymbol{r}$ can be written as

$$\boldsymbol{r}(t) = \boldsymbol{r}_1 \cos\omega t + \boldsymbol{r}_2 \sin\omega t. \tag{5.168}$$

Vectors $\boldsymbol{R}_1$, $\boldsymbol{R}_2$, $\boldsymbol{r}_1$, $\boldsymbol{r}_2$ are time-independent. Substitution of (5.165) and (5.168) into the equation of motion (5.2) leads to two simultaneous matrix equations for the unknowns $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$,

$$\begin{aligned} (\boldsymbol{K} - \omega^2 \boldsymbol{M})\boldsymbol{r}_1 + \omega \boldsymbol{C}\, \boldsymbol{r}_2 &= \boldsymbol{R}_1, \\ -\omega \boldsymbol{C}\, \boldsymbol{r}_1 + (\boldsymbol{K} - \omega^2 \boldsymbol{M})\boldsymbol{r}_2 &= \boldsymbol{R}_2. \end{aligned} \tag{5.169}$$

The monitored quantities in a steady harmonic vibration are the resonance phenomena. It is therefore essential to introduce damping into the computation. As a consequence, the response of the structure $\boldsymbol{r}$ always possesses both phases, even when the applied loads act in phase. In that case it only holds that $\boldsymbol{R}_2 = \boldsymbol{O}$.

5.4.1 Direct solution in complex numbers

As noted above, both response parts $\boldsymbol{r}_1$, $\boldsymbol{r}_2$ need to be computed, which makes the approach differ from the usual FEM algorithms. On the other hand, the Fortran language, which is most often used to program the FEM, is by default equipped with complex arithmetics. This can be used to advantage.

The load vector $\boldsymbol{R}$ and the displacement vector $\boldsymbol{r}$ can be expressed as real parts of complex vectors. It holds that

$$\boldsymbol{R}(t) = \mathrm{Re}\left[\widehat{\boldsymbol{R}}\, \mathrm{e}^{i\omega t}\right], \quad \boldsymbol{r}(t) = \mathrm{Re}\left[\widehat{\boldsymbol{r}}\, \mathrm{e}^{i\omega t}\right], \tag{5.170}$$

where

$$\widehat{\boldsymbol{R}}(t) = \boldsymbol{R}_1 - \mathrm{i}\, \boldsymbol{R}_2, \quad \widehat{\boldsymbol{r}}(t) = \boldsymbol{r}_1 - \mathrm{i}\, \boldsymbol{r}_2. \tag{5.171}$$

One can verify the correctness of the first relation (5.170) by substituting from (5.171). We obtain

$$\boldsymbol{R}(t) = \mathrm{Re}\left[(\boldsymbol{R}_1 - \mathrm{i}\, \boldsymbol{R}_2)\,(\cos \omega t + \mathrm{i} \sin \omega t)\right]. \tag{5.172}$$

After modification,

$$\boldsymbol{R}(t) = \mathrm{Re}\left[(\boldsymbol{R}_1 \cos \omega t + \boldsymbol{R}_2 \sin \omega t) + \mathrm{i}(\boldsymbol{R}_1 \sin \omega t - \boldsymbol{R}_2 \cos \omega t)\right], \tag{5.173}$$

which is identical to (5.165). Correctness of the second relation (5.170) can be proved in a similar fashion.

If the vector $\widehat{\boldsymbol{R}}$ is shown in the complex plane, the relation between the amplitude a_n, the phase shift φ_n and the components R_{1n}, R_{2n} becomes obvious. It is clear that they represent two different ways to write a complex number (see Fig. 5.6).

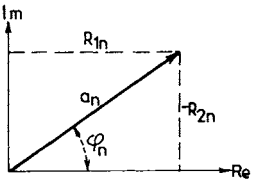

Figure 5.6: The relationship between amplitude, phase shift and parts of a complex number

It is sufficient to satisfy (5.2) at each time instant for both $\boldsymbol{R}(t)$ and $\boldsymbol{r}(t)$, to make $\widehat{\boldsymbol{R}}$ and $\widehat{\boldsymbol{r}}$ comply with

$$\left(-\omega^2 \boldsymbol{M} + \mathrm{i}\omega \boldsymbol{C} + \boldsymbol{K}\right) \widehat{\boldsymbol{r}} = \widehat{\boldsymbol{R}}. \tag{5.174}$$

The equation (5.174) is formally identical to the equilibrium condition in a static solution. However, the matrix $\boldsymbol{K}$ is replaced by the complex matrix

$$\widehat{\boldsymbol{K}} = \left(-\omega^2 \boldsymbol{M} + \mathrm{i}\omega \boldsymbol{C} + \boldsymbol{K}\right), \tag{5.175}$$

which is called the *dynamic stiffness*. The algorithm of the solution is the same as in statics, but all operations must be done in complex arithmetics.[13]

5.4.2 Mode decomposition method

Let us assume in what follows that we know the eigenfrequencies and the modes of undamped eigenvibration. The eigenmodes are collected in the matrix $\boldsymbol{Y}$ such that the individual modes constitute the columns of $\boldsymbol{Y}$. It is additionally assumed that they are normalized with respect to the mass matrix [compare with (5.20)]:

$$\boldsymbol{Y}^T \boldsymbol{M}\, \boldsymbol{Y} = \boldsymbol{I}. \tag{5.176}$$

The approximate solution is sought in the form of a linear combination of p lowest eigenmodes

$$\widehat{\boldsymbol{r}} = \boldsymbol{Y}\, \boldsymbol{q}, \tag{5.177}$$

where $\boldsymbol{q}$ is the vector of unknown complex coefficients. Substitution of (5.177) into (5.174) gives

$$\left(-\omega^2 \boldsymbol{M} + \mathrm{i}\omega \boldsymbol{C} + \boldsymbol{K}\right) \boldsymbol{Y}\, \boldsymbol{q} = \widehat{\boldsymbol{R}}. \tag{5.178}$$

Multiplication of (5.178) from the left by $\boldsymbol{Y}^T$ leads (with the orthogonality conditions applied) to the system of linear algebraic equations with complex coefficients

$$\left(-\omega^2 \boldsymbol{I} + \mathrm{i}\omega \boldsymbol{Y}^T \boldsymbol{C} \boldsymbol{Y} + \boldsymbol{\Omega}^2\right) \boldsymbol{q} = \boldsymbol{Y}^T \widehat{\boldsymbol{R}}, \tag{5.179}$$

where $\boldsymbol{\Omega}^2$ is a diagonal matrix, whose terms are the squares of the circular eigenfrequencies.

Proportional damping

The system obtained for a general damping matrix $\boldsymbol{C}$ consists of simultaneous equations, and the following section is devoted to its solution. In many practical cases, the damping can be considered proportional to mass and stiffness, so that

$$\boldsymbol{Y}^T \boldsymbol{C} \boldsymbol{Y} = 2\boldsymbol{\Omega}_b, \tag{5.180}$$

where $\boldsymbol{\Omega}_b$ is a diagonal matrix with terms [compare with (5.114)]

$$\omega_{bk} = \xi_k \omega_k, \tag{5.181}$$

ω_k are the eigenfrequencies, and ξ_k are the coefficients of relative damping.

The system (5.179) decouples into

$$(-\omega^2 + 2\mathrm{i}\omega\omega_k\xi_k + \omega_k^2) q_k = \boldsymbol{y}_k^T \widehat{\boldsymbol{R}} \tag{5.182}$$

[13] It is appropriate to note that the matrix of dynamic stiffness $\widehat{\boldsymbol{K}}$ is a function of the forcing frequency ω. It is necessary to repeat the computation for each frequency.

with the solution

$$q_k = \boldsymbol{y}_k^T \widehat{\boldsymbol{R}} \frac{1}{-\omega^2 + 2\mathrm{i}\omega\omega_k\xi_k + \omega_k^2}. \tag{5.183}$$

Using (5.177) and (5.183) the relationship between the vector of output quantities (displacements, internal forces, stresses, etc.) and the load vector can be formally written with matrix $\boldsymbol{G}$ in the form

$$\boldsymbol{v} = \boldsymbol{G}\boldsymbol{Y}\boldsymbol{Q}\boldsymbol{Y}^T\widehat{\boldsymbol{R}} = \boldsymbol{U}\widehat{\boldsymbol{R}}, \tag{5.184}$$

where $\boldsymbol{Q}$ is a diagonal matrix with complex terms

$$Q_{ii} = \frac{1}{-\omega^2 + 2\mathrm{i}\omega\omega_k\xi_k + \omega_k^2}. \tag{5.185}$$

Equation (5.184) holds also for more than one right-hand side; only the vectors $\widehat{\boldsymbol{R}}$ and $\boldsymbol{v}$ are replaced by matrices $\boldsymbol{R}$ and $\boldsymbol{V}$.

The only matrix that changes with different frequencies ω in equation (5.184) is the matrix $\boldsymbol{Q}$, whose terms can be for different ω recomputed from (5.185). For the real and imaginary part of $\boldsymbol{Q}$ we have

$$\mathrm{Re}\,[Q_{jj}] = \frac{\omega_j^2 - \omega^2}{\left(\omega_j^2 - \omega^2\right)^2 + 4\,\omega^2\omega_j^2\xi_j^2}, \tag{5.186}$$

$$\mathrm{Im}\,[Q_{jj}] = \frac{-2\,\omega\omega_j\xi_j}{\left(\omega_j^2 - \omega^2\right)^2 + 4\,\omega^2\omega_j^2\xi_j^2}. \tag{5.187}$$

Equation (5.184) can then be rewritten without using complex numbers as two equations for two phases of the output quantities

$$\mathrm{Re}[\boldsymbol{v}] = \boldsymbol{G}\boldsymbol{Y}\left(\mathrm{Re}[\boldsymbol{Q}]\boldsymbol{Y}^T\boldsymbol{R}_1 + \mathrm{Im}[\boldsymbol{Q}]\boldsymbol{Y}^T\boldsymbol{R}_2\right), \tag{5.188}$$

$$\mathrm{Im}[\boldsymbol{v}] = \boldsymbol{G}\boldsymbol{Y}\left(-\mathrm{Re}[\boldsymbol{Q}]\boldsymbol{Y}^T\boldsymbol{R}_2 + \mathrm{Im}[\boldsymbol{Q}]\boldsymbol{Y}^T\boldsymbol{R}_1\right). \tag{5.189}$$

Vectors (or matrices, for more right-hand sides) $\boldsymbol{Y}^T\boldsymbol{R}_1$ and $\boldsymbol{Y}^T\boldsymbol{R}_2$ can be computed beforehand, independently of given frequencies.

The algorithm for the solution of forced vibration by mode decomposition is therefore very simple for the case of proportional damping. It can be included in any computer program which is able to compute the eigenmodes normalized with respect to the mass matrix.[14]

Nonproportional damping

Interaction of the structure and the subgrade is a typical example of a nonproportional damping. The damping in the upper part of the system differs considerably from the damping of the foundation. This is due partly to different material properties, partly to the dissipation of energy into the semi-infinite half-space. Therefore, it is not always possible to accept the damping model of (5.180). There are two options with respect to the mode decomposition method:

[14]The algorithm can also be used for an analytic solution by the Koloušek exact deflection method. The eigenmodes are normalized with respect to the mass μ (mass per unit length of the beam), which means $\int \mu w_i w_j ds = \delta_{ij}$.

- Consider the mode decomposition method only as a means for reducing the dimension of the problem, and solve (5.179) according to Section 5.4.1, or
- transform (5.179) into a system of independent equations.

The first alternative does not need a detailed explanation. The second one, on the other hand, requires solution to a *problem of damped eigenvibration.* This is rather a demanding task for larger systems, and the majority of programs does not include this algorithm. There is a simple approximate solution based on the condensation by undamped eigenmodes $\boldsymbol{Y}$, besides the Lanczos method (Section 5.2.11). The nonproportional damping leads to the homogeneous system of algebraic equations

$$\boldsymbol{I}\ddot{\boldsymbol{q}}_t + \widehat{\boldsymbol{C}}\dot{\boldsymbol{q}}_t + \boldsymbol{\Omega}^2\boldsymbol{q}_t = \boldsymbol{O}, \tag{5.190}$$

where

$$\widehat{\boldsymbol{C}} = \boldsymbol{Y}^T\boldsymbol{C}\boldsymbol{Y}.$$

The vectors $\boldsymbol{q}_t$ are characterized not only by amplitude, but also by phase for the nonproportional damping case. If p modes of undamped eigenvibration are used to reduce the dimension to an approximate system (5.190), is is necessary to solve $2p$ equations with real coefficients. Each eigenmode is thus characterized by the vector of amplitudes and the vector of phases. This is the consequence of the fact that the nonproportionally damped structure has no stationary nodes of vibration. The method has been proposed in [83]. It is based on the simple idea that the p original equations are complemented by additional p equations

$$\dot{\boldsymbol{q}}_t = \dot{\boldsymbol{q}}_t. \tag{5.191}$$

Equations (5.190) and (5.191) can be written in the matrix form as

$$\begin{bmatrix} \boldsymbol{O} & \boldsymbol{I} \\ \boldsymbol{I} & \widehat{\boldsymbol{C}} \end{bmatrix} \begin{Bmatrix} \ddot{\boldsymbol{q}}_t \\ \dot{\boldsymbol{q}}_t \end{Bmatrix} + \begin{bmatrix} -\boldsymbol{I} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{\Omega}^2 \end{bmatrix} \begin{Bmatrix} \dot{\boldsymbol{q}}_t \\ \boldsymbol{q}_t \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{O} \\ \boldsymbol{O} \end{Bmatrix}. \tag{5.192}$$

The inverse matrix to the first square matrix in (5.192) is

$$\begin{bmatrix} \boldsymbol{O} & \boldsymbol{I} \\ \boldsymbol{I} & \widehat{\boldsymbol{C}} \end{bmatrix}^{-1} = \begin{bmatrix} -\widehat{\boldsymbol{C}} & \boldsymbol{I} \\ \boldsymbol{I} & \boldsymbol{O} \end{bmatrix}. \tag{5.193}$$

Thus (5.192) can be transformed by using (5.193) into

$$\begin{bmatrix} \widehat{\boldsymbol{C}} & \boldsymbol{\Omega}^2 \\ \boldsymbol{I} & \boldsymbol{O} \end{bmatrix} \begin{Bmatrix} \dot{\boldsymbol{q}}_t \\ \boldsymbol{q}_t \end{Bmatrix} = \lambda \begin{Bmatrix} \dot{\boldsymbol{q}}_t \\ \boldsymbol{q}_t \end{Bmatrix}. \tag{5.194}$$

The terms of the square matrix of (5.194) are real, but the matrix is nonsymmetric. If the damping is subcritical (which is the case for building structures), the solution of (5.194) gives $2p$ eigenmodes with eigenvalues λ. The eigenvalues are complex conjugate with a negative real part. The imaginary part represents the circular frequencies of the damped vibration. The numerical solution of the eigenvalue problem (5.194) can be obtained by using standard algorithms from scientific subroutine libraries. The authors have used subprograms of the library SSP (Subroutine Scientific Package), marketed by IBM. The system matrix (5.194) is full; its size, however, permits the whole computation to be done in-core.

The computed damped vibration eigenmodes of (5.194) can be used to transform (5.179) to a canonical form even for nonproportional damping. Because this holds also for time-dependent loading, it is advantageous to start from equations of motion written as a system of first-order ordinary differential equations. It holds that

$$\boldsymbol{A}\dot{\boldsymbol{z}} + \boldsymbol{B}\boldsymbol{z} = \boldsymbol{f}, \tag{5.195}$$

where

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{O} & \boldsymbol{I} \\ \boldsymbol{I} & \widehat{\boldsymbol{C}} \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} -\boldsymbol{I} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{\Omega}^2 \end{bmatrix},$$

$$\boldsymbol{f} = \begin{Bmatrix} \boldsymbol{O} \\ \boldsymbol{Y}^T\boldsymbol{R} \end{Bmatrix}, \quad \boldsymbol{z} = \begin{Bmatrix} \dot{\boldsymbol{q}}_t \\ \boldsymbol{q}_t \end{Bmatrix}.$$

Similarly to the preceding section, the vectors $\boldsymbol{f}$, $\boldsymbol{z}$ can be described as

$$\boldsymbol{f}(t) = \mathrm{Re}\left[\widehat{\boldsymbol{f}}\,\mathrm{e}^{i\omega t}\right], \quad \boldsymbol{z}(t) = \mathrm{Re}\left[\widehat{\boldsymbol{z}}\,\mathrm{e}^{i\omega t}\right]. \tag{5.196}$$

The equations of motion after substitution of (5.196) into (5.195) become

$$(\mathrm{i}\omega\boldsymbol{A} + \boldsymbol{B})\,\widehat{\boldsymbol{z}} = \widehat{\boldsymbol{f}}. \tag{5.197}$$

Now, the transformation into eigenmodes is repeated once again. The transformation can be written as

$$\widehat{\boldsymbol{z}} = \boldsymbol{\Delta}\boldsymbol{c}, \tag{5.198}$$

where $\boldsymbol{\Delta}$ is the matrix of coefficients of the *damped vibration eigenmodes* and $\boldsymbol{c}$ is the vector of coefficients of the linear combination. Substitution of (5.198) into (5.197) and multiplication from the left by $\boldsymbol{\Delta}^T$ gives

$$(\mathrm{i}\omega\boldsymbol{\mathcal{A}} + \boldsymbol{\mathcal{B}})\,\boldsymbol{c} = \boldsymbol{\Delta}^T\widehat{\boldsymbol{f}}, \tag{5.199}$$

where

$$\boldsymbol{\mathcal{A}} = \boldsymbol{\Delta}^T\boldsymbol{A}\boldsymbol{\Delta}, \quad \boldsymbol{\mathcal{B}} = \boldsymbol{\Delta}^T\boldsymbol{B}\boldsymbol{\Delta}.$$

As the eigenmodes of the damped system are orthogonal, we have that the matrices $\boldsymbol{\mathcal{A}}$, $\boldsymbol{\mathcal{B}}$ are diagonal. Therefore (5.199) is a system of linearly independent algebraic equations with complex coefficients. Components of $\boldsymbol{c}$ are given by

$$c_j = \frac{\left(\boldsymbol{\Delta}^T\widehat{\boldsymbol{f}}\right)_j}{(\mathrm{i}\omega - \lambda_j)\,\mathcal{A}_{jj}}. \tag{5.200}$$

Finally, after some manipulations, we obtain

$$\boldsymbol{r} = \boldsymbol{Y}\,\mathrm{Re}\left[\boldsymbol{\Delta}\boldsymbol{c}e^{i\omega t}\right]. \tag{5.201}$$

If the solution produces not only modes for generalized displacements, but also for stresses (internal forces), then the expression $\mathrm{Re}\left[\boldsymbol{\Delta}\boldsymbol{c}e^{i\omega t}\right]$ can be used to compute the corresponding mechanical quantities.

Part II

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Chapter 6

Semianalytical Methods

Engineering structures often have constant geometric and material properties along a certain direction. Typical examples are prismatic plates and box girders, frequently used in bridge engineering. From the numerical point of view, bridges also have favorable boundary conditions—they are simply supported at both ends, and box structures are usually stiffened by end diaphragms. In addition to line structures, axially symmetric structures also fall within the category of structures that can be efficiently solved by a combination of the FEM with Fourier series expansions. Such an approach is called a semianalytical method. It was first applied by Grafton and Strom in [65] to the solution of axially symmetric shells. The semianalytical approach was later extended to axially symmetric bodies in [178], and to prismatic folded plates in [45] and [46]. A general formulation was given in [75] and [127]. In this chapter, we derive the relations needed for the solution of rectangular plates, and we briefly describe the general formulation for curved folded plates.

The semianalytical method transforms the solution of a two-dimensional problem into the solution of a sequence of one-dimensional problems, and the solution of a three-dimensional problem into the solution of a sequence of two-dimensional problems. Examples of two-dimensional structures given in Fig. 6.1 show that the structure is not divided into elements but into strips. This is the origin of the frequently used term finite strip method.

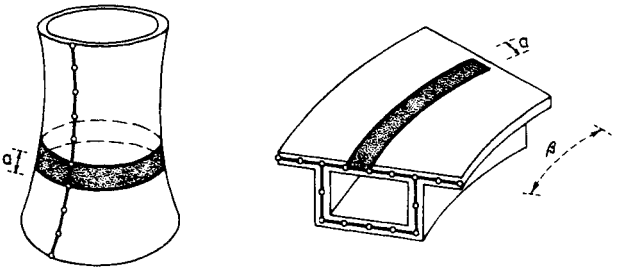

Figure 6.1: Examples of two-dimensional structures

6.1 Energy-based beam analysis by Fourier series

The fundamental idea of the approach based on Fourier series is illustrated by the solution of bending of a simply supported beam (Fig. 6.2). The total potential energy is given by

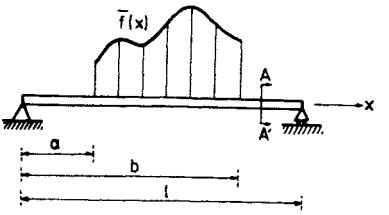

Figure 6.2: Simply supported beam

$$\Pi(w) = \frac{1}{2}\int_0^l EI\left(\frac{d^2w}{dx^2}\right)^2 dx - \int_a^b \overline{f} w\, dx. \qquad (6.1)$$

The boundary conditions requiring

$$w = 0, \quad \frac{d^2w}{dx^2} = 0 \quad \text{for } x = 0 \text{ and for } x = l.$$

are satisfied by the Fourier series

$$w(x) = \sum_{n=1}^{\infty} {}^n w \sin\frac{n\pi x}{l}.$$

Application of Fourier series is based on the orthogonality of functions $\sin\frac{n\pi x}{l}$ on the interval $< 0, l >$, i.e., on the property

$$\int_0^l \sin\frac{n\pi x}{l} \sin\frac{m\pi x}{l}\, dx = \begin{cases} \frac{l}{2} & \text{for } n = m, \\ 0 & \text{for } n \neq m. \end{cases} \qquad (6.2)$$

Same as the displacement function w, the load function $\overline{f}$ can also be represented by a Fourier series

$$\overline{f}(x) = \sum_{n=1}^{\infty} {}^n\overline{f} \sin\frac{n\pi x}{l}. \qquad (6.3)$$

The evaluation of the coefficients ${}^n\overline{f}$ is made easy by the orthogonality property (6.2). Multiplying expression (6.3) by $\sin\frac{m\pi x}{l}$ and integrating from 0 to l (scalar product) we get

$$\int_0^l \overline{f}(x) \sin\frac{m\pi x}{l}\, dx = \sum_{n=1}^{\infty} {}^n\overline{f} \int_0^l \sin\frac{n\pi x}{l} \sin\frac{m\pi x}{l}\, dx.$$

Due to (6.2), this relation leads to

$${}^n\overline{f} = \frac{2}{l}\int_0^l \overline{f}(x) \sin\frac{n\pi x}{l}\, dx.$$

Substituting into (6.1) and evaluating the integrals we obtain

$$\Pi = \frac{EI}{2}\frac{l}{2}\sum_{n=1}^{\infty}\left(\frac{n\pi}{l}^4\right)({}^n w)^2 - \frac{l}{2}\sum_{n=1}^{\infty} {}^n\overline{f}\, {}^n w.$$

The coefficients ${}^n w$ can now be determined from the condition of minimum potential energy, which gives

$$\frac{\partial \Pi}{\partial\, {}^n w} = 0 \;\Rightarrow\; {}^n w = \frac{{}^n\overline{f}}{EI}\left(\frac{l}{n\pi}\right)^4.$$