An SDE whose left-hand side is linear in both dependent and independent variables is known as a linear equation of first order. For example,

$$\frac{dy}{dx} + yP = Q \tag{13.6}$$

is a linear equation of first order, where y is a stochastic dependent variable, Q is a forcing function, P is a sink function, and x is an independent variable. Functions P and Q may be stochastic. Another example of a linear equation is

$$\frac{dy}{dx} + xy = 5x \tag{13.7}$$

In contrast, the equation

$$\frac{dy}{dx} + xy^2 = 5x \tag{13.8}$$

is not a linear equation.

A given system may receive input at time t = 0 that is not deterministic. In ordinary differential equations, the initial condition(s) may be random variables. In a partial differential equation, it may be specified as a random process. Depending upon the properties of the system, the uncertain input may be further propagated or it may be dissipated. Given sufficient data, the problem is to find the probability distribution of the system output. However, at times, enough information may not be available to determine the complete distribution and one may have to be content with only the first few moments of it. In all of these situations SDEs arise. Depending on the way randomness is considered, stochastic differential equations can also be classified as (i) differential equations with random initial conditions, (ii) differential equations with random forcing functions, (iii) differential equations with random boundary conditions, (iv) differential equations with random coefficients, (v) differential equations with random geometrical domains, and (vi) differential equations that combine two or more of these conditions. A solution of an SDE is a stochastic process that satisfies it. Because the dependent variable is stochastic, the concepts of mean square continuity, stochastic differentiation, and stochastic integration are invoked. These concepts define the continuity, differentiation, and integration of a stochastic process.

13.4 Fundamental Concepts in Solving SDEs

Since the solution of an SDE is in terms of stochastic variable(s), the concepts such as continuity, differentiation, and integration are modified and defined to take stochasticity into account. These concepts are introduced in what follows. The concept of mean square continuity is useful in the study of stochastic processes. A process $X(t)$ is said to be continuous in mean square sense if it satisfies the condition

$$E[X(t+\tau)-X(t)]^2 \to 0 \text{ as } \tau \to 0 \tag{13.9}$$

where $\tau > 0$ is the time lag or delay. Expanding Eq. 13.9, one obtains

$$E[X(t+\tau)-X(t)]^2 = E[X(t+\tau)^2 - E[X(t+\tau)X(t)] - E[X(t)X(t+\tau)] + E[X(t)^2] \tag{13.10}$$

The right-hand side of this equation approaches zero as $\tau \to 0$. Clearly, the process is continuous if $E[X(t_1)X(t_2)]$ is continuous along the time axis. This implies that

$$E[X(t+\tau)] \to E[X(t)] \text{ as } \tau \to 0 \tag{13.11}$$

A related concept in differentiation is of mean square derivative of a stochastic process. A process has mean square derivative at t if the following limit is satisfied in the mean square sense:

$$\lim_{\tau \to 0} E\left[\frac{X(t+\tau)-X(t)}{\tau} - \frac{dX(t)}{dt}\right]^2 = 0 \tag{13.12}$$

13.4.1 Stochastic Differentiation

The mean square derivative is useful because its properties can be represented in terms of the second-order properties of the stochastic process, that is, the covariance function. A stationary stochastic process $X(t)$ is differentiable in the mean square sense if its autocorrelation function $R(\tau)$ is differentiable up to the second order. The derivative of the expected value of $X(t)$ is equal to the expected value of the derivative of $X(t)$. This property can be generalized to an nth derivative if it exists. If $X(t)$ is nonstationary then it is differentiable in the mean square if the second-order partial derivative of its autocorrelation function $R(t_1, t_2)$ with respect to t_1 and t_2 [i.e., $\partial^2 R(t_1,t_2)/\partial t_1 \partial t_2$], exists at $t_1 = t_2$. Similarly, a stochastic process $X(t)$ is nth-order differentiable if $\partial^{2n} R(t_1,t_2)/\partial t_1^n \partial t_2^n$ exists at $t_1 = t_2$.

13.4.2 Stochastic Integration

A mean-square integral of a stochastic process involves the limit of the sum in the mean square sense. Thus, a stochastic process $X(t)$ is integrable if

$$\int_a^b X(t)dt = \lim_{\Delta t_i \to 0} \sum_i X(t_i)\Delta t_i \tag{13.13}$$

exists. Similar to stochastic differentiation, $X(t)$ is integrable over the interval (a, b) if the double integral of the autocorrelation function is bounded:

$$\int_a^b \int_a^b |R_X(t_1, t_2)|\, dt_1\, dt_2 < \infty \tag{13.14}$$

The condition for the existence of a mean-square derivative can also be considered from the spectral representation, which is an integral in the mean square sense. The derivative of $X(t)$ can be expressed as

$$W(t) = \frac{dX}{dt} = \int_{-\infty}^{\infty} e^{i\omega t} i\omega \, dZ_X(\omega) = \int_{-\infty}^{\infty} e^{i\omega t} dZ_W(\omega) \tag{13.15}$$

Here $X(t)$ is considered to be a zero-mean stochastic process, ω represents angular frequency, and $dZ_W(\omega)$ represents the Fourier amplitude of the stochastic process. If the derivative $W(t)$ is stationary, then

$$dZ_W = i\omega \, dZ_X \tag{13.16}$$

The spectrum of the derivative can then be expressed as

$$S_{WW}(\omega) = \omega^2 S_{XX}(\omega) \tag{13.17}$$

The covariance function of the derivative W can be expressed as

$$R_{WW}(\tau) = \int_{-\infty}^{\infty} e^{i\omega\tau} S_{WW}(\omega) d\omega = \int_{-\infty}^{\infty} e^{i\omega\tau} \omega^2 S_{XX}(\omega) \tag{13.18a}$$

$$d\omega = -\frac{d^2 R_{XX}}{d\tau^2} \tag{13.18b}$$

For $\tau = 0$, the variance of the derivative follows:

$$\sigma_W{}^2 = \int_{-\infty}^{\infty} \omega^2 S_{XX}(\omega) d\omega = -\frac{d^2 R_{XX}}{d\tau^2}\Big|_{\tau=0} < \infty \tag{13.19}$$

Since

$$W(t) = \frac{dX(t)}{dt}$$

the covariance functions of X and W are related as

$$\frac{d^2 R_{XX}(\tau)}{d\tau^2} = -R_{WW}(\tau) \tag{13.20}$$

by Eq. 13.18. However, Eq. 13.17 shows a simple algebraic relation between their spectra:

$$S_{XX}(\omega) = \frac{S_{WW}(\omega)}{\omega^2} \tag{13.21}$$

Thus, for X to be stationary, its variance must be finite:

$$\sigma_X{}^2 = \int_{-\infty}^{\infty} \frac{S_{WW}(\omega)}{\omega^2} d\omega < \infty \tag{13.22}$$

Using Eq. 13.13, one can write

$$E[|\int_a^b X(t)dt|]^2 = \int_a^b \int_a^b R_X(t_1, t_2)dt_1 dt_2 \tag{13.23}$$

The order of integration and expectation is interchangeable. To illustrate it, consider a process $Y(t)$ as

$$Y(t) = \int_0^t X(s)ds \tag{13.24}$$

Then

$$E[Y(t)] = \mu_Y(t) = E[\int_0^t X(s)ds] = \int_0^t E[X(s)]ds = \int_0^t \mu_X(s)ds \tag{13.25}$$

The property can be extended to obtain the correlation function as follows:

$$\begin{aligned} E[Y(t_1)Y(t_2)] = R_Y(t_1, t_2) &= E[(\int_0^{t_1} X(t)dt)(\int_0^{t_2} X(s)ds)] \\ &= E[\int_0^{t_1}\int_0^{t_2} X(t)X(s)dtds] \\ &= \int_0^{t_1}\int_0^{t_2} E[X(t)X(s)]dtds = \int_0^{t_1}\int_0^{t_2} R_X(t, s)dtds \end{aligned} \tag{13.26}$$

In a similar manner, the autocovariance of a stochastic integral can be determined. Thus,

$$\begin{aligned} \mathrm{cov}_Y(t_1, t_2) &= R_Y(t_1, t_2) - \mu_Y(t_1)\mu_Y(t_2) \\ &= \int_0^{t_1}\int_0^{t_2} R_X(t, s)dtds - \int_0^{t_1}\int_0^{t_2} \mu_X(t)\mu_X(s)\,dtds \end{aligned} \tag{13.27}$$

If $t_1 = t_2 = t$ is inserted in Eq. 13.27, the result is the variance of the stochastic integral:

$$\begin{aligned} \sigma_Y{}^2(t) &= \mathrm{cov}_Y(t, t) = R_Y(t,t) - \mu_Y{}^2(t) \\ &= \int_0^t\int_0^t R_X(z,s)dz\,ds - \int_0^t\int_0^t \mu_X(z)\mu_X(s)dz\,ds \end{aligned} \tag{13.28}$$

$$\rho_Y(t_1,t_2)=\frac{\int_0^{t_1}\int_0^{t_2} R_X(t,\,s)dtds-\int_0^{t_1}\int_0^{t_2}\mu_X(t)\mu_X(s)dtds}{\sqrt{\int_0^{t_1}\int_0^{t_1} R_X(z,\,s)dzds-\int_0^{t_1}\int_0^{t_1}\mu_X(z)\mu_X(s)dz\,ds}}\times\frac{1}{\int_0^{t_2}\int_0^{t_2} R_X(z,\,s)dzds-\int_0^{t_2}\int_0^{t_2}\mu_X(t)\mu_X(s)dz\,ds} \tag{13.29}$$

Now, let us look at some examples. The first example treats outflow as a random function.

Example 13.1 The water level in a lake in India during the summer months of no rainfall is governed by the following differential equation:

$$K\frac{\partial H}{\partial t}=-Q$$

where Q is discharge from the lake through an outlet and K is a parameter. The water level in the lake is expressed as

$$H(t)=H_0(t)+h(t)$$

where H_0 is a deterministic function of time and $h(t)$ is a random process with $E[h(t)]=0$ and with the autocorrelation function represented as $R_h(t_1,t_2)=C_V{}^2(K)\exp[-a\,|\,t_2-t_1\,|\,]$, where a is the correlation time parameter and $C_V{}^2(K)$ is the coefficient of variation of K. Determine the mean, the autocorrelation function, and the covariance function of discharge.

Solution The differential equation is expressed as

$$Q(t)=-K\left\{\frac{\partial}{\partial t}[H_0+h(t)]\right\}$$

Taking the expectation gives

$$E[Q(t)]=-E\{K\frac{\partial}{\partial t}[H_0+h(t)]\}=-K\frac{\partial H_0}{\partial t}-KE\left[\frac{\partial}{\partial t}h(t)\right]$$

Assuming the stochastic process mean-square continuity, we can extend Eq. 13.11 and Eq. 13.12 to the derivative of the stochastic process as

$$E\left[X^{'}(t)\right]=\frac{dE\left[X(t)\right]}{dt}$$

We then have

$$KE\left[\frac{\partial}{\partial t}h(t)\right] = K\frac{dE[h(t)]}{dt} = 0$$

Thus, we get

$$E[Q(t)] = -E\{K\frac{\partial}{\partial t}[H_0 + h(t)]\} = -K\frac{\partial H_0}{\partial t}$$

The autocorrelation function of discharge can be expressed as

$$\begin{aligned}
E[Q(t_1)Q(t_2)] &= R_Q(t_1,t_2) = E\{[-K\frac{\partial H_0(t_1)}{\partial t_1} - K\frac{\partial h(t_1)}{\partial t_1}][-K\frac{\partial H_0(t_2)}{\partial t_2} - K\frac{\partial h(t_2)}{\partial t_2}]\} \\
&= E[K^2\frac{\partial H_0(t_1)}{\partial t_1}\frac{\partial H_0(t_2)}{\partial t_2} + K^2\frac{\partial H_0(t_1)}{\partial t_1}\frac{\partial h(t_2)}{\partial t_2} + K^2\frac{\partial H_0(t_2)}{\partial t_2}\frac{\partial h(t_1)}{\partial t_1} + K^2\frac{\partial h(t_1)}{\partial t_1}\frac{\partial h(t_2)}{\partial t_2}] \\
&= K^2E[\frac{\partial H_0(t_1)}{\partial t_1}\frac{\partial H_0(t_2)}{\partial t_2}] + K^2\frac{\partial^2}{\partial t_1 \partial t_2}E\left[h(t_1)h(t_2)\right] \\
&= K^2\frac{\partial^2}{\partial t_1 \partial t_2}[R_{H_0}(t_1,t_2)] + K^2\frac{\partial^2}{\partial t_1 \partial t_2}[R_{h_0}(t_1,t_2)] \\
&= K^2\frac{\partial^2}{\partial t_1 \partial t_2}[R_{H_0}(t_1,t_2)] + K^2\frac{\partial^2}{\partial t_1 \partial t_2}[C_V{}^2\exp(-a\,|\,t_2 - t_1\,|)] \\
&= K^2\frac{\partial^2}{\partial t_1 \partial t_2}[R_{H_0}(t_1,t_2)] - a^2K^2C_V^2(K)\exp\left(-a\,|\,t_2 - t_1\,|\right)
\end{aligned}$$

The variance is obtained by inserting $t_1 = t_2 = t$, which yields

$$\sigma_Q{}^2(t) = R_Q(t,t) - \mu_Q{}^2(t) = K^2\left[\frac{\partial H_0(t)}{\partial t}\right]^2 + K^2a^2C_V{}^2 - K^2\left(\frac{\partial H_0}{\partial t}\right)^2 = K^2a^2C_V{}^2$$

The case of random initial condition is exemplified next.

Example 13.2 A linear differential equation

$$\frac{dS}{dt} = -Q\,,\ \ S = KQ$$

is frequently used for stream base-flow recession. Here S is the storage in a watershed at time t, Q is discharge, and K is the residence time. The initial condition is the following: At $t = 0$, $Q(0) = Q_0$. It is assumed that Q_0 is a random variable with mean μ_{Q_0} and variance $\sigma^2_{Q_0}$. Determine the solution of the differential equation and the mean μ and the variance, covariance, and autocorrelation function of discharge Q.

Solution The differential equation can be recast as

$$\frac{dQ}{dt}+\frac{1}{K}Q=0$$

Its solution is

$$Q(t)=Q_0\exp(-t/K)$$

Here Q_0 is a random variable. Therefore $Q(t)$ is a stochastic process comprising a family of exponential recessions with random initial value Q_0.

The mean of $Q(t)$ is

$$E[Q(t)]=E[Q_0e^{-t/K}]=E[Q_0]E[e^{-t/K}]=\mu_{Q_0}e^{-t/K}$$

The autocorrelation function of Q(t) is obtained as

$$\begin{aligned}E[Q(t_1)Q(t_2)]=R_Q(t_1,t_2)&=E[(Q_0e^{-t_1/K})(Q_0e^{-t_2/K})]=E[Q_0{}^2]e^{-(t_1+t_2)/K}\\&=(\sigma_{Q_0}^2+\mu_{Q_0}^2)\exp[-(t_1+t_2)/K]\end{aligned}$$

The covariance function of Q(t) is given as

$$\begin{aligned}\operatorname{cov}(t_1,t_2)&=R_Q(t_1,t_2)-\mu_Q(t_1)\mu_Q(t_2)\\&=(\sigma_{Q_0}^2+\mu_{Q_0}^2)\exp[-(t_1+t_2)/K]-\mu_{Q_0}^2\exp[-(t_1+t_2)/K]\\&=\sigma_{Q_0}^2\exp[-(t_1+t_2)/K]\end{aligned}$$

The variance of $Q(t)$ is obtained by setting $t_1=t_2$, which gives

$$\sigma_Q^2=R_Q(t,t)-\mu_Q^2(t)=\sigma_{Q_0}^2\exp[-2t/K]$$

The correlation coefficient of $Q(t)$ is

$$\rho_Q(t_1,t_2)=\frac{\sigma_{Q_0}^2\exp[-(t_1+t_2)/K]}{\sigma_{Q_0}^2\sqrt{\exp(-2t_1/K)}\sqrt{\exp(-2t_2/K)}}=1$$

Taking K as 24 hours, mean Q_0 as 10 m^3/s^2, and variance as 68 m^3/s^2, we can plot $E[Q(t)]$ and the variance, covariance, and autocorrelation function of $Q(t)$. The results are presented in Fig. 13-1.

Now consider the case where the input is random.

Example 13.3 A surface runoff hydrograph from an area represented by a linear reservoir can be described mathematically as

$$\frac{dQ(t)}{dt}+AQ(t)=AP(t)=P_*(t),\;\; Q(0)=0$$

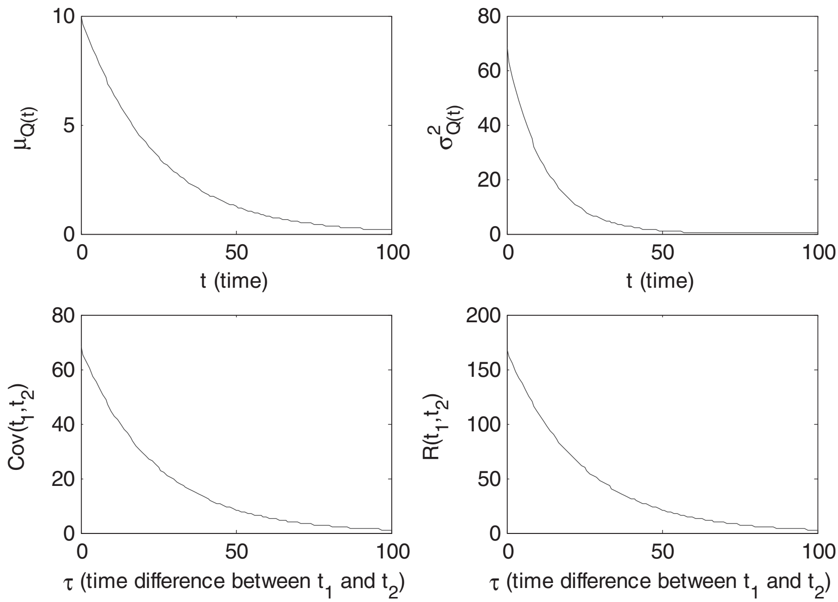

Figure 13-1 *The expectation, variance, covariance and autocorrelation functions of* Q(t).

where $P(t)$ is rainfall intensity, Q is unit surface runoff hydrograph, A is a reservoir coefficient, and t is time. It is assumed that A is a random variable uniformly distributed as

$$f(A) = \frac{1}{A_u - A_L}$$

with A_L = lower limit of A and A_u = upper limit, and that $P(t)$ is a stochastic process expressed as $P_* = A^2 \exp(-At)$. Determine the mean, variance, covariance, autocorrelation function, and the coefficient of correlation of Q. This problem is discussed by Lin and Wang (1996).

Solution The solution of the differential equation is

$$Q(t) = \int_0^t P_*(s)\exp[-A(t-s)]ds = \int_0^t A^2 \exp(-As)\exp[-A(t-s)]ds$$
$$= A^2 t \exp(-At)$$

The stochastic process $Q(t)$ now has an explicit solution. Its mean is expressed as

$$E[Q(t)] = \mu_Q(t) = E[A^2 t \exp(-At)] = \int_{A_L}^{A_u} A^2 t \exp(-At)\frac{1}{(A_u - A_L)} dA$$
$$= \frac{\left[(A_L t + 1)^2 + 1\right]\exp(-A_L t) - \left[(A_u t + 1)^2 + 1\right]\exp(-A_u t)}{t^2 (A_u - A_L)}$$

$$R_Q(t_1,t_2) = E\left[Q(t_1)Q(t_2)\right] = E\left[A^4 t_1 t_2 \exp\left(-At_1 - At_2\right)\right]$$
$$= \int_{A_L}^{A_u} A^4 t_1 t_2 \exp\left(-At_1 - At_2\right)\frac{1}{A_u - A_L}\,dA$$
$$= \frac{-B_1 \exp\left[-A_u(t_1+t_2)\right] + B_2 \exp[-A_L(t_1+t_2)]}{t_1^4 t_2^4\left(A_u - A_L\right)}$$

Letting $t' = t_1 + t_2$, we get

$$B_1 = A_u^4 t'^4 + 4A_u^3 t'^3 + 12A_u^2 t'^2 + 24A_u t' + 24$$
$$B_2 = A_L^4 t'^4 + 4A_L^3 t'^3 + 12A_L^2 t'^2 + 24A_L t' + 24$$

The covariance is obtained as

$$\mathrm{cov}(t_1,t_2) = R_Q(t_1,t_2) - \mu_Q(t_1)\mu_Q(t_2)$$
$$= \frac{-B_1 \exp\left[-A_u(t_1+t_2)\right] + B_2 \exp[-A_L(t_1+t_2)]}{t_1^4 t_2^4\left(A_u - A_L\right)} -$$
$$\frac{\left[C_1 \exp(-A_L t_1) - C_2 \exp(-A_u t_1)\right]\left[C_3 \exp(-A_L t_2) - C_4 \exp(-A_u t_2)\right]}{t_1^2 t_2^2 (A_u - A_L)^2}$$

$$C_1 = 2 + 2A_L t_1 + A_L^2 t_1^2;\ C_2 = 2 + 2A_u t_1 + A_u^2 t_1^2$$
$$C_3 = 2 + 2A_L t_2 + A_L^2 t_2^2;\ C_4 = 2 + 2A_u t_2 + A_u^2 t_2^2$$

The variance is obtained by setting $t_1 = t_2$ as

$$\sigma_Q^2 = R_Q(t,t) - \mu_Q^2(t)$$
$$= \frac{1}{A_u - A_L}\left[\frac{B_3 \exp(-2A_L t) - B_4 \exp(-2A_u t)}{4t^3}\right] - \frac{\left[B_5 \exp(-A_L t) - B_6 \exp(-A_u t)\right]^2}{(A_u - A_L)^2 t^4}$$

$$B_3 = 3 + 6A_L t + 6A_L^2 t^2 + 4A_L^3 t^3 + 2A_L^4 t^4$$
$$B_4 = 3 + 6A_u t + 6A_u^2 t^2 + 4A_u^3 t^3 + 2A_u^4 t^4$$
$$B_5 = A_L^2 t^2 + 2A_L t + 2$$
$$B_6 = A_u^2 t^2 + 2A_u t + 2$$

The correlation of Q is

$$\rho_Q(t_1,t_1) = \frac{\mathrm{cov}\left[t_1,t_2\right]}{\sigma_Q^2(t_1)\sigma_Q^2(t_2)} = 1$$

Taking the lower and upper limits of A as 0.10 and 0.5 hour^{-1}, respectively, we can plot the mean, variance, covariance function, and autocorrelation function of $Q(t)$. The results are given in .com.

The next two examples deal with the cases where a parameter in the IUH is a random variable.

Example 13.4 A watershed is represented by a cascade of n equal reservoirs, each with reservoir coefficient k considered as a random variable. This representation is referred to as the Nash cascade and is popularly used for modeling surface runoff. The IUH of this cascade is

$$h_n = \frac{k}{(n-1)!}(kt)^{n-1}e^{-kt}$$

Because k is a random variable, h_n is the stochastic IUH of the n-reservoir cascade. Determine the mean, variance, and the first three moments of h_n (or IUH). Assume that k has a normal distribution with mean μ_k and variance σ_k^2. This watershed problem was discussed by Lin and Wang (1996). Take the mean of k as 2.14, σ_k as 0.25, and $n = 3$. Plot the computed functions of $h_n(t)$.

Solution The mth moment of h_n can be expressed as

$$E[h_n^m(t)] = E[\frac{k}{(n-1)!}(tk)^{n-1}e^{-kt}]^m = \int_{-\infty}^{\infty}[\frac{k}{(n-1)!}(kt)^{n-1}e^{-kt}]^m f_k(k)dk$$

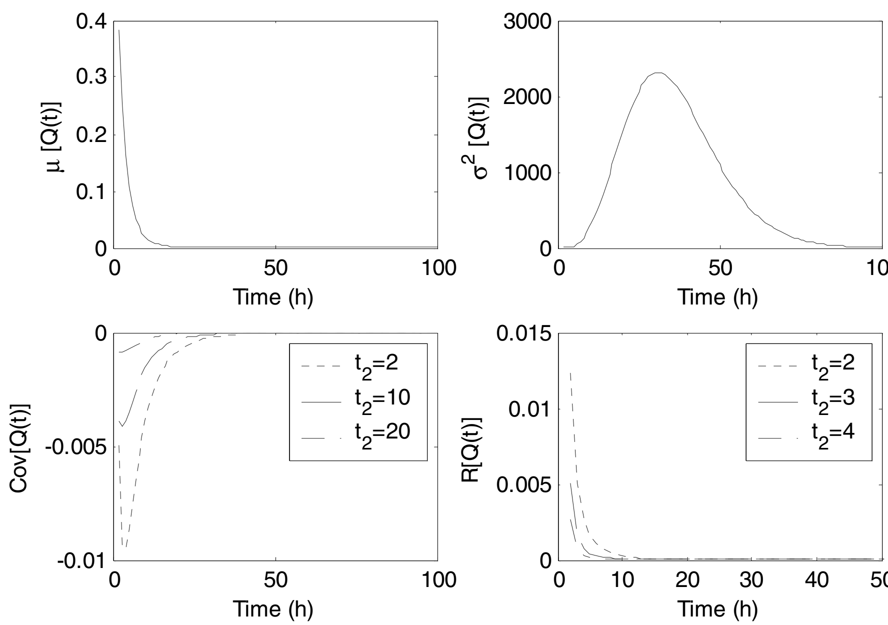

Figure 13-2 The expectation, variance, covariance and autocorrelation functions of Q(t).