vector $\mathbb{V} = (V_1, \ldots, V_{m_\mathbb{V}})$. Consequently, the $\mathbb{R}^m$-valued random vector $\mathbf{U}$ can be written as

$$\mathbf{U} = \mathbf{h}(\mathbb{V}), \quad \mathbf{U}^{\text{obs}} = \mathbf{h}^{\text{obs}}(\mathbb{V}), \quad \mathbf{U}^{\text{nobs}} = \mathbf{h}^{\text{nobs}}(\mathbb{V}), \tag{8.2}$$

in which $v \mapsto \mathbf{h}(v) = (\mathbf{h}^{\text{obs}}(v), \mathbf{h}^{\text{nobs}}(v))$ is a deterministic nonlinear transformation from $\mathbb{R}^{m_\mathbb{V}}$ into $\mathbb{R}^m = \mathbb{R}^{m_{\text{obs}}} \times \mathbb{R}^{m_{\text{nobs}}}$ which can be constructed solving the discretized boundary value problem.

(3) Experimental data sets

It is assumed that ν_{exp} experimental data sets are available for the observation vector $\mathbf{U}^{\text{obs}}$. Each experimental data set corresponds to partial experimental data (only the trace of the displacement field on Γ_{obs} is observed) with a limited length (ν_{exp} is relatively small). These ν_{exp} experimental data sets correspond to measurements of ν_{exp} experimental configurations associated with the same boundary value problem. For configuration ℓ, with $\ell = 1, \ldots, \nu_{\text{exp}}$, the observation vector (corresponding to $\mathbf{U}^{\text{obs}}$ for the computational model) is denoted by $\mathbf{u}^{\text{exp},\ell}$ and belongs to $\mathbb{R}^m$. Therefore, the available data are made up of the ν_{exp} vectors $\mathbf{u}^{\text{exp},1}, \ldots, \mathbf{u}^{\text{exp},\nu_{\text{exp}}}$ in $\mathbb{R}^m$. Below, it is assumed that $\mathbf{u}^{\text{exp},1}, \ldots, \mathbf{u}^{\text{exp},\nu_{\text{exp}}}$ can be viewed as ν_{exp} independent realizations of a random vector $\mathbf{U}^{\text{exp}}$ defined on a probability space $(\Theta^{\text{exp}}, \mathcal{T}^{\text{exp}}, \mathcal{P}^{\text{exp}})$ and corresponding to random observation vector $\mathbf{U}^{\text{obs}}$ (but noting that random vectors $\mathbf{U}^{\text{exp}}$ and $\mathbf{U}^{\text{obs}}$ are not defined on the same probability space).

(4) Stochastic inverse problem to be solved

The problem to be solved concerns the identification of the unknown non-Gaussian random vector $\mathbb{V}$ representing the spatial discretization of fourth-order tensor-valued random field $\{\mathbb{C}(\mathbf{x}), \mathbf{x} \in \Omega\}$. Such an identification is carried out using partial and limited experimental data $\mathbf{u}^{\text{exp},1}$, $\ldots$, $\mathbf{u}^{\text{exp},\nu_{\text{exp}}}$ relative to the random observation vector $\mathbf{U}^{\text{obs}}$ such that $\mathbf{U}^{\text{obs}} = \mathbf{h}^{\text{obs}}(\mathbb{V})$ in which $\mathbf{h}^{\text{obs}}$ is a given deterministic nonlinear mapping. The components of the random vector $\mathbf{U}^{\text{nobs}}$, such that $\mathbf{U}^{\text{nobs}} = \mathbf{h}^{\text{nobs}}(\mathbb{V})$ in which $\mathbf{h}^{\text{nobs}}$ is a given deterministic nonlinear mapping, are not used for the identification but will be used for performing the quality assessment of the identification .

8.2 Construction of a family of prior algebraic probability models (PAPM) for the tensor-valued random field in elasticity theory

The notations introduced in Section 8.1 are used. We are interested in constructing a family of prior algebraic probability models (PAPM) for the non-Gaussian fourth-order tensor-valued random field $\{\mathbb{C}(\mathbf{x}), \mathbf{x} \in \Omega\}$ defined on a probability space $(\Theta, \mathcal{T}, \mathcal{P})$, in which $\mathbb{C}(\mathbf{x}) = \{\mathbb{C}_{ijk\ell}(\mathbf{x})\}_{ijk\ell}$.

8.2.1 Construction of the tensor-valued random field $\mathbb{C}$

The mean value $\{\underline{\mathbb{C}}(\mathbf{x}), \mathbf{x} \in \Omega\}$ of random field $\{\mathbb{C}(\mathbf{x}), \mathbf{x} \in \Omega\}$ is assumed to be given (this mean value can also be considered as unknown in the context of the inverse problem for the identification of the model parameters as proposed in Step 2 of Section 8.3). It is a deterministic tensor-valued field $\mathbf{x} \mapsto \{\underline{\mathbb{C}}_{ijkh}(\mathbf{x})\}_{ijkh}$ such that

$$E\{\mathbb{C}_{ijkh}(\mathbf{x})\} = \underline{\mathbb{C}}_{ijkh}(\mathbf{x}), \quad \forall \mathbf{x} \in \Omega, \tag{8.3}$$

where E is the mathematical expectation. For instance, in the context of the linear elasticity of a heterogeneous microstructure, tensor-valued function $\underline{\mathbb{C}}$ would be chosen as the mean model of a random anisotropic elastic microstructure at the mesoscale. It should be noted that the known symmetries, such as orthotropic symmetry or transversally isotropic symmetry, can be taken into account with the mean model represented by tensor $\{\underline{\mathbb{C}}_{ijkh}(\mathbf{x})\}_{ijkh}$. Nevertheless, in Section 8.2, the random fluctuations $\{\mathbb{C}_{ijkh}(\mathbf{x}) - \underline{\mathbb{C}}_{ijkh}(\mathbf{x})\}_{ijkh}$ around the mean tensor-valued field will be assumed to be purely anisotropic, without any symmetries. For instance, probability models for the elasticity tensor-valued random field with uncertain material symmetries are analyzed in (Guilleminot and Soize 2010). Below, we present an extension of the theory developed in (Soize 2006; 2008b). The family of prior tensor-valued random field $\mathbb{C}(\mathbf{x})$ is then constructed as

$$\mathbb{C}(\mathbf{x}) = C^0(\mathbf{x}) + \mathbf{C}(\mathbf{x}).$$

The deterministic tensor-valued field $C^0(\mathbf{x}) = \{C^0_{ijkh}(\mathbf{x})\}_{ijkh}$ will be called the "deterministic lower-bound tensor-valued field" which will be symmetric and positive definite. The tensor-valued random field $\mathbf{C}(\mathbf{x}) = \mathbf{C}_{ijkh}(\mathbf{x})\}_{ijkh}$ defined by $\mathbf{C}(\mathbf{x}) = \mathbb{C}(\mathbf{x}) - C^0(\mathbf{x})$ will be called the "fluctuations tensor-valued random field". This tensor will be almost surely symmetric and positive-definite. Tensor-valued field $C^0(\mathbf{x})$ should be such that the mean value $\underline{\mathbf{C}}(\mathbf{x}) = E\{\mathbf{C}(\mathbf{x})\} = \underline{\mathbb{C}}(\mathbf{x}) - C^0(\mathbf{x})$ is symmetric and positive definite. Below, we present the entire construction and we give the corresponding main mathematical properties.

(i)- Mathematical notations. In order to study the mathematical properties of the tensor-valued random field $\mathbb{C}(\mathbf{x})$, we introduce the real Hilbert space $\mathcal{H} = \{\, \mathbf{u} = (u_1, u_2, u_3)\,,\, u_j \in L^2(\Omega)\,\}$ equipped with the inner product

$$<\mathbf{u}\,,\mathbf{v}>_{\mathcal{H}}= \int_\Omega <\mathbf{u}(\mathbf{x})\,,\mathbf{v}(\mathbf{x})> \,d\mathbf{x},$$

and with the associated norm $\|\mathbf{u}\|_{\mathcal{H}} \;=< \mathbf{u}\,,\mathbf{u} >_{\mathcal{H}}^{1/2}$, in which $<\mathbf{u}(\mathbf{x})\,,\mathbf{v}(\mathbf{x})> \,=\, u_1(\mathbf{x})v_1(\mathbf{x}) + u_2(\mathbf{x})v_2(\mathbf{x}) + u_3(\mathbf{x})v_3(\mathbf{x})$ and where $L^2(\Omega)$ denotes the set of all the square integrable functions from Ω into $\mathbb{R}$. Let $\mathcal{V} \subset \mathcal{H}$ be the real Hilbert space representing the set of admissible displacement fields with values in $\mathbb{R}^3$ such that

$$\mathcal{V} = \{\mathbf{u} \in \mathcal{H}\ ,\ \partial\mathbf{u}/\partial x_1\,,\, \partial\mathbf{u}/\partial x_2\,,\, \partial\mathbf{u}/\partial x_3 \text{ in } \mathcal{H}\ ,\ \mathbf{u} = 0 \text{ on } \Gamma_0\},$$

equipped with the inner product

$$<\mathbf{u}\,,\mathbf{v}>_{\mathcal{V}} = <\mathbf{u}\,,\mathbf{v}>_{\mathcal{H}} + <\frac{\partial\mathbf{u}}{\partial x_j}\,,\frac{\partial\mathbf{v}}{\partial x_j}>_{\mathcal{H}}$$

and with the associated norm $\|\mathbf{u}\|_{\mathcal{V}} \;=< \mathbf{u}\,,\mathbf{u} >_{\mathcal{V}}^{1/2}$. The convention of summation over repeated latin indices is used. Let $L^2(\Theta, \mathcal{V})$ be the space of all the second-order random variable $\mathbf{U} = \{\mathbf{U}(\mathbf{x}), \mathbf{x} \in \Omega\}$ defined on $(\Theta, \mathcal{T}, \mathcal{P})$ with values in $\mathcal{V}$, equipped with the inner product

$$\ll \mathbf{U}, \mathbf{V} \gg = E\{<\mathbf{U}\,,\mathbf{V}>_{\mathcal{V}}\}$$

and the associated norm $|||\mathbf{U}||| = \ll \mathbf{U}, \mathbf{U} \gg^{1/2}$. For all $\mathbf{U}$ in $L^2(\Theta, \mathcal{V})$, we then have

$$|||\mathbf{U}|||^2 = E\{\|\mathbf{U}\|_{\mathcal{V}}^2\} < +\infty.$$

Finally, the operator norm $\|\mathbb{T}\|$ of any fourth-order tensor $\mathbb{T} = \{\mathbb{T}_{ijkh}\}_{ijkh}$ is defined by $\|\mathbb{T}\| = \sup_{\|z\|_F \le 1} \|\mathbb{T} : z\|_F$ in which $z = \{z_{kh}\}_{kh}$ is a second-order tensor such that $\|z\|_F^2 = z_{kh}\, z_{kh}$ and where $\{\mathbb{T} : z\}_{ij} = \mathbb{T}_{ijkh} z_{kh}$.

(ii)- Mean tensor-valued field $\underline{\mathbb{C}}(\mathbf{x})$. We introduce the deterministic bilinear form $\underline{c}(\mathbf{u}, \mathbf{v})$ related to the mean tensor-valued field $\underline{\mathbb{C}}$,

$$\underline{c}(\mathbf{u}, \mathbf{v}) = \int_\Omega \underline{\mathbb{C}}_{ijkh}(\mathbf{x})\, \varepsilon_{kh}(\mathbf{u})\ \varepsilon_{ij}(\mathbf{v})\, d\mathbf{x}, \tag{8.4}$$

in which the second-order strain tensor $\{\varepsilon_{kh}\}_{kh}$ is such that

$$\varepsilon_{kh}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_h} + \frac{\partial u_h}{\partial x_k}\right).$$

For all $\mathbf{x}$, the fourth-order real tensor $\underline{\mathbb{C}}(\mathbf{x}) = \{\underline{\mathbb{C}}_{ijhk}(\mathbf{x})\}_{ijhk}$ of the elastic coefficients verifies the usual property of symmetry

$$\underline{\mathbb{C}}_{ijkh}(\mathbf{x}) = \underline{\mathbb{C}}_{jikh}(\mathbf{x}) = \underline{\mathbb{C}}_{ijhk}(\mathbf{x}) = \underline{\mathbb{C}}_{khij}(\mathbf{x}), \tag{8.5}$$

and for all symmetric second-order real tensor $z = \{z_{kh}\}_{kh}$, tensor $\underline{\mathbb{C}}(\mathbf{x})$ verifies the following property,

$$b_0\, z_{kh}\, z_{kh} \;\leq\; \underline{\mathbb{C}}_{ijkh}(\mathbf{x}) z_{kh} z_{ij} \;\leq\; b_1\, z_{kh}\, z_{kh}, \tag{8.6}$$

in which b_0 and b_1 are deterministic positive constants which are independent of $\mathbf{x}$. Taking into account Eqs. (8.5) and (8.6), it can be deduced that bilinear form $\underline{\mathfrak{c}}(\mathbf{u},\mathbf{v})$ is symmetric, positive-definite, continuous on $\mathcal{V}\times\mathcal{V}$ and is elliptic on $\mathcal{V}$, that is to say, is such that

$$\underline{\mathfrak{c}}(\mathbf{u},\mathbf{u}) \geq k_0\, \|\mathbf{u}\|_{\mathcal{V}}^2. \tag{8.7}$$

Equation (8.7) can easily be proven using Eq. (8.4), Eq. (8.6) and the Korn inequality which is written as $\int_\Omega \varepsilon_{kh}(\mathbf{u})\, \varepsilon_{kh}(\mathbf{u})\, d\mathbf{x} \geq b_2\, \|\mathbf{u}\|_{\mathcal{V}}^2$. It can then be deduced that $k_0 = b_0\, b_2$ is a positive constant.

(iii)- Deterministic lower-bound tensor-valued field $C^0(\mathbf{x})$. The deterministic lower-bound tensor-valued field $\{C^0(\mathbf{x}), \mathbf{x}\in\Omega\}$ is a given deterministic field which is introduced to guaranty the ellipticity condition of the tensor-valued random field $\mathbb{C}(\mathbf{x})$. We will give two examples for the construction of $C^0(\mathbf{x})$. The fourth-order real tensor $C^0_{ijkh}(\mathbf{x})$ must verify the usual property of symmetry (similarly to Eq. (8.5)) and for all symmetric second-order real tensor $\{z_{kh}\}_{kh}$ must be such that

$$b_0^0\, z_{kh}\, z_{kh} \;\leq\; C^0_{ijkh}(\mathbf{x}) z_{kh}\, z_{ij} \;\leq\; b_1^0\, z_{kh}\, z_{kh}, \quad \forall \mathbf{x}\in\Omega, \tag{8.8}$$

in which b_0^0 and b_1^0 are deterministic positive constants which are independent of $\mathbf{x}$. Let $\underline{\mathbf{C}}(\mathbf{x})$ be the tensor-valued deterministic field defined by

$$\underline{\mathbf{C}}(\mathbf{x}) = \underline{\mathbb{C}}(\mathbf{x}) - C^0(\mathbf{x}), \quad \forall \mathbf{x}\in\Omega. \tag{8.9}$$

In addition, tensor-valued field C^0 must be constructed for that, for all $\mathbf{x}$, tensor $\underline{\mathbf{C}}(\mathbf{x})$, which verifies the symmetry property (see Eq. (8.5)), must be positive definite, that is to say, for all non zero symmetric second-order real tensor $\{z_{kh}\}_{kh}$, must be such that

$$\underline{\mathbf{C}}_{ijkh}(\mathbf{x})\, z_{kh}\, z_{ij} > 0, \quad \forall \mathbf{x}\in\Omega. \tag{8.10}$$

From Eqs. (8.9), (8.6) and (8.8), it can easily be deduced that

$$\underline{\mathbf{C}}_{ijkh}(\mathbf{x})\, z_{kh}\, z_{ij} \;\leq\; b_1^1\, z_{kh}\, z_{kh}, \quad \forall \mathbf{x} \in \Omega, \tag{8.11}$$

in which $b_1^1 = b_1 - b_0^0 > 0$ is a positive finite constant independent of $\mathbf{x}$. Introducing the deterministic bilinear form $\underline{\mathbb{c}}^0(\mathbf{u},\mathbf{v})$ related to the deterministic lower-bound tensor-valued field $C^0(\mathbf{x})$,

$$\underline{\mathbb{c}}^0(\mathbf{u},\mathbf{v}) = \int_\Omega C^0_{ijkh}(\mathbf{x})\, \varepsilon_{kh}(\mathbf{u})\; \varepsilon_{ij}(\mathbf{v})\, d\mathbf{x}, \tag{8.12}$$

it can be shown, as previously, that this bilinear form is symmetric, positive-definite, continuous on $\mathcal{V} \times \mathcal{V}$ and is elliptic on $\mathcal{V}$, that is to say, is such that

$$\underline{\mathbb{c}}^0(\mathbf{u},\mathbf{u}) \geq k_0^0\, \|\mathbf{u}\|^2_{\mathcal{V}}, \tag{8.13}$$

in which $k_0^0 = b_0^0\, b_2$ is a positive constant.

Example 1. In certain cases, a deterministic lower bound $C^{\min}$ independent of $\mathbf{x}$ can be constructed for a given microstructure (Guilleminot et al. 2011). The fourth-order tensor $C^{\min}$ is symmetric and positive definite. For all $\mathbf{x}$ in Ω, we then have $C^0(\mathbf{x}) = C^{\min}$.

Example 2. If there is no available information to construct the deterministic lower-bound tensor-valued field $\{C^0(\mathbf{x}), \mathbf{x} \in \Omega\}$, we can define it as $C^0(\mathbf{x}) = \epsilon_0\, \underline{\mathbf{C}}(\mathbf{x})$ in which $0 < \epsilon_0 < 1$ can be chosen as small as one wants. With such a choice, we have $\underline{\mathbf{C}}(\mathbf{x}) = (1-\epsilon_0)\underline{\mathbb{C}}(\mathbf{x})$.

(iv)- Random fluctuations tensor-valued field $\mathbf{C}(\mathbf{x})$. The random fluctuations tensor-valued field $\{\mathbf{C}(\mathbf{x}), \mathbf{x} \in \Omega\}$ is defined on probability space $(\Theta, \mathcal{T}, \mathcal{P})$. In (Soize 2006; 2008b), the random fluctuations tensor-valued field $\{\mathbf{C}(\mathbf{x}), \mathbf{x} \in \Omega\}$ is constructed in order that all the following properties listed below be verified.
For all $\mathbf{x}$ fixed in Ω, the fourth-order real tensor $\mathbf{C}_{ijkh}(\mathbf{x})$ is symmetric,

$$\mathbf{C}_{ijkh}(\mathbf{x}) = \mathbf{C}_{jikh}(\mathbf{x}) = \mathbf{C}_{ijhk}(\mathbf{x}) = \mathbf{C}_{khij}(\mathbf{x}), \tag{8.14}$$

and is positive definite, that is to say, for all non zero symmetric second-order real tensor $\{z_{kh}\}_{kh}$, we have,

$$\mathbf{C}_{ijkh}(\mathbf{x})\, z_{kh}\, z_{ij} > 0. \tag{8.15}$$

The mean function of random field $\{\mathbf{C}(\mathbf{x}), \mathbf{x} \in \Omega\}$ is equal to the tensor-valued deterministic field $\{\underline{\mathbf{C}}(\mathbf{x}), \mathbf{x} \in \Omega\}$ defined by Eq. (8.9),

$$E\{\mathbf{C}(\mathbf{x})\} = \underline{\mathbf{C}}(\mathbf{x}), \quad \forall \mathbf{x} \in \Omega. \tag{8.16}$$

Let $\mathbf{c}(\mathbf{U}, \mathbf{V})$ be the random bilinear form defined by

$$\mathbf{c}(\mathbf{U}, \mathbf{V}) = \int_\Omega \mathbf{C}_{ijkh}(\mathbf{x})\, \varepsilon_{kh}(\mathbf{U})\ \varepsilon_{ij}(\mathbf{V})\, d\mathbf{x}. \tag{8.17}$$

The available information used to construct the random field $\mathbf{C}$ implies (see (Soize 2006)) that the bilinear form $(\mathbf{U}, \mathbf{V}) \mapsto E\{\mathbf{c}(\mathbf{U}, \mathbf{V})\}$ is symmetric, positive definite, continuous on $L^2(\Theta, \mathcal{V}) \times L^2(\Theta, \mathcal{V})$, is not elliptic but is such that, for all $\mathbf{U}$ in $L^2(\Theta, \mathcal{V})$, we have

$$\sqrt{E\{\mathbf{c}(\mathbf{U}, \mathbf{U})^2\}} \geq k_1\, E\{\|\mathbf{U}\|^2_{\mathcal{V}}\}, \tag{8.18}$$

in which k_1 is a positive constant. Equation (8.18) implies that the following elliptic boundary value problem $E\{\mathbf{c}(\mathbf{U}, \mathbf{V})\} = E\{f(\mathbf{V})\}$ for all $\mathbf{V}$ in $L^2(\Theta, \mathcal{V})$, in which $f(\mathbf{v})$ is a given continuous linear form on $\mathcal{V}$, has a unique random solution $\mathbf{U}$ in $L^2(\Theta, \mathcal{V})$, but the random solution $\mathbf{U}$ is not a continuous function of the parameters.

(v)- Prior algebraic probability model (PAPM) for the tensor-valued random field $\mathbb{C}(\mathbf{x})$. The non-Gaussian fourth-order tensor-valued random field $\{\mathbb{C}(\mathbf{x}), \mathbf{x} \in \Omega\}$ is defined on probability space $(\Theta, \mathcal{T}, \mathcal{P})$ and such that, for all $\mathbf{x}$ in Ω,

$$\mathbb{C}(\mathbf{x}) = C^0(\mathbf{x}) + \mathbf{C}(\mathbf{x}), \quad \forall \mathbf{x} \in \Omega, \tag{8.19}$$

in which the deterministic lower-bound tensor-valued field $\{C^0(\mathbf{x}), \mathbf{x} \in \Omega\}$ is defined in (iii) and where the random fluctuations tensor-valued field $\{\mathbf{C}(\mathbf{x}), \mathbf{x} \in \Omega\}$ is defined in (iv). Let $\mathbb{c}(\mathbf{U}, \mathbf{V})$ be the random bilinear form defined by

$$\mathbb{c}(\mathbf{U}, \mathbf{V}) = \int_\Omega \mathbb{C}_{ijkh}(\mathbf{x})\, \varepsilon_{kh}(\mathbf{U})\ \varepsilon_{ij}(\mathbf{V})\, d\mathbf{x}, \tag{8.20}$$

and let $c(\mathbf{U}, \mathbf{V})$ be the bilinear form defined by

$$c(\mathbf{U}, \mathbf{V}) = E\{\mathbb{c}(\mathbf{U}, \mathbf{V})\}. \tag{8.21}$$

Then, it can easily be verified that the bilinear form $c(\mathbf{U}, \mathbf{V})$ is symmetric, positive-definite, continuous on $L^2(\Theta, \mathcal{V}) \times L^2(\Theta, \mathcal{V})$ and is elliptic, that is to say, for all $\mathbf{U}$ in $L^2(\Theta, \mathcal{V})$, we have

$$c(\mathbf{U}, \mathbf{U}) \geq k_0^0\, |||\mathbf{U}|||^2. \tag{8.22}$$

Equation (8.22) implies that the following elliptic boundary value problem $E\{\mathbb{c}(\mathbf{U}, \mathbf{V})\} = E\{f(\mathbf{V})\}$ for all $\mathbf{V}$ in $L^2(\Theta, \mathcal{V})$, in which $f(\mathbf{v})$ is a given continuous linear form on $\mathcal{V}$, has a unique random solution $\mathbf{U}$ in $L^2(\Theta, \mathcal{V})$ and the random solution $\mathbf{U}$ is a continuous function of the parameters.

8.2.2 Construction of the tensor-valued random field $\mathbf{C}$

In this section, we summarize the construction of random field $\{\mathbf{C}(\mathbf{x}), \mathbf{x} \in \Omega\}$ whose available information and properties have been defined in Section 8.2.1-(iv) and for which the details of this construction can be found in (Soize 2008b) and (Soize 2006). For all $\mathbf{x}$ fixed in $\mathcal{I} = \{\mathbf{x}^1, \ldots, \mathbf{x}^{N_p}\} \subset \Omega$, the fourth-order tensor-valued random variable $\mathbf{C}(\mathbf{x})$ is represented by a real random matrix $[\mathbf{A}(\mathbf{x})]$. Let I and J be the new indices belonging to $\{1, \ldots, 6\}$ such that $I = (i, j)$ and $J = (k, h)$ with the following correspondence: $1 = (1,1), 2 = (2,2), 3 = (3,3), 4 = (1,2), 5 = (1,3)$ and $6 = (2,3)$. Thus, for all $\mathbf{x}$ in Ω, the random (6×6) real matrix $[\mathbf{A}(\mathbf{x})]$ is such that

$$[\mathbf{A}(\mathbf{x})]_{IJ} = \mathbf{C}_{ijkh}(\mathbf{x}), \quad \mathbf{x} \in \Omega. \tag{8.23}$$

For all $\mathbf{x}$ fixed in Ω, due to the symmetry and positive-definiteness properties (defined by Eqs. (8.14) and (8.15)) of the random fourth-order tensor $\mathbf{C}(\mathbf{x})$, it can be deduced that $[\mathbf{A}(\mathbf{x})]$ is a random variable with values in the set $\mathbb{M}_6^+(\mathbb{R})$ of all the (6×6) real symmetric positive-definite matrices. The $\mathbb{M}_6^+(\mathbb{R})$-valued random field $\{[\mathbf{A}(\mathbf{x})], \mathbf{x} \in \Omega\}$, indexed by Ω, defined on the probability space $(\Theta, \mathcal{T}, \mathcal{P})$, is constituted of $6 \times (6+1)/2 = 21$ mutually dependent real-valued random fields defining the fourth-order tensor-valued random field $\mathbf{C}$ indexed by Ω. Let $\mathbf{x} \mapsto [\underline{a}(\mathbf{x})]$ be the random field from Ω into $\mathbb{M}_6^+(\mathbb{R})$ defined by

$$[\underline{a}(\mathbf{x})]_{IJ} = \underline{\mathbf{C}}_{ijkh}(\mathbf{x}), \quad \mathbf{x} \in \Omega, \tag{8.24}$$

in which the random field $\mathbf{x} \mapsto \underline{\mathbf{C}}(\mathbf{x})$ is defined by Eq. (8.9). Consequently, the mean function of random field $[\mathbf{A}]$ is such that

$$E\{[\mathbf{A}(\mathbf{x})]\} = [\underline{a}(\mathbf{x})], \quad \mathbf{x} \in \Omega. \tag{8.25}$$

Since $[\underline{a}(\mathbf{x})]$ belongs to $\mathbb{M}_6^+(\mathbb{R})$, there is an upper triangular invertible matrix $[\underline{L}(\mathbf{x})]$ in $\mathbb{M}_6(\mathbb{R})$ (the set of all the (6×6) real matrices) such that

$$[\underline{a}(\mathbf{x})] = [\underline{L}(\mathbf{x})]^T \, [\underline{L}(\mathbf{x})], \quad \mathbf{x} \in \Omega. \tag{8.26}$$

From Eq. (8.11), $\mathbf{x} \mapsto [\underline{a}(\mathbf{x})]$ is a bounded function on Ω and it can then be assumed that $\mathbf{x} \mapsto [\underline{L}(\mathbf{x})]$ is also bounded function on Ω. For all $\mathbf{x}$ fixed in Ω, the random matrix $[\mathbf{A}(\mathbf{x})]$ can be written as

$$[\mathbf{A}(\mathbf{x})] = [\underline{L}(\mathbf{x})]^T \, [\mathbf{G}_0(\mathbf{x})] \, [\underline{L}(\mathbf{x})], \tag{8.27}$$

in which $\mathbf{x} \mapsto [\mathbf{G}_0(\mathbf{x})]$ is a random field defined on $(\Theta, \mathcal{T}, \mathcal{P})$, indexed by $\mathbb{R}^3$, with values in $\mathbb{M}_6^+(\mathbb{R})$, such that for all $\mathbf{x}$ in $\mathbb{R}^3$

$$E\{[\mathbf{G}_0(\mathbf{x})]\} = [\, I_6], \tag{8.28}$$

in which $[\,I_6]$ is the unity matrix. The random field $[\mathbf{G}_0]$ is completely defined below.

(i)- Probability model of the random field $[\mathbf{G}_0]$. The random field $\mathbf{x} \mapsto [\mathbf{G}_0(\mathbf{x})]$ is constructed as a homogeneous and normalized non-Gaussian positive-definite matrix-valued random field, defined on probability space $(\Theta, \mathcal{T}, \mathcal{P})$, indexed by $\mathbb{R}^3$, with values in $\mathbb{M}_6^+(\mathbb{R})$. This random field is constructed as a non-linear mapping of 21 independent second-order centered homogeneous Gaussian random fields $\mathbf{x} \mapsto U_{jj'}(\mathbf{x})$, $1 \leq j \leq j' \leq 6$, defined on probability space $(\Theta, \mathcal{T}, \mathcal{P})$, indexed by $\mathbb{R}^3$, with values in $\mathbb{R}$, and named the stochastic germs of the non-Gaussian random field $[\mathbf{G}_0]$.

(i.1)- Random fields $U_{jj'}$ as the stochastic germs of the random field $[\boldsymbol{G}_0]$. The stochastic germs are constituted of 21 independent second-order centered homogeneous Gaussian random fields $\mathbf{x} \mapsto U_{jj'}(\mathbf{x})$, $1 \leq j \leq j' \leq 6$, defined on probability space $(\Theta, \mathcal{T}, \mathcal{P})$, indexed by $\mathbb{R}^3$, with values in $\mathbb{R}$ and such that

$$E\{U_{jj'}(\mathbf{x})\} = 0, \quad E\{U_{jj'}(\mathbf{x})^2\} = 1. \tag{8.29}$$

Consequently, all these random fields are completely and uniquely defined by the 21 autocorrelation functions $R_{U_{jj'}}(\boldsymbol{\zeta}) = E\{U_{jj'}(\mathbf{x} + \boldsymbol{\zeta})\, U_{jj'}(\mathbf{x})\}$ defined for all $\boldsymbol{\zeta} = (\zeta_1, \zeta_2, \zeta_3)$ in $\mathbb{R}^3$ and such that $R_{U_{jj'}}(0) = 1$. In order to obtain a class having a reasonable number of parameters, these autocorrelation functions are written as $R_{U_{jj'}}(\boldsymbol{\zeta}) = \rho_1^{jj'}(\zeta_1) \times \rho_2^{jj'}(\zeta_2) \times \rho_3^{jj'}(\zeta_3)$ in which, for all $k = 1, 2, 3$, one has $\rho_k^{jj'}(0) = 1$ and for all $\eta_k \neq 0$,

$$E\rho_k^{jj'}(\zeta_k) = 4(L_k^{jj'})^2/(\pi^2\zeta_k^2)\ \sin^2\Big(\pi\zeta_k/(2L_k^{jj'})\Big)\,. \tag{8.30}$$

in which $L_1^{jj'}, L_2^{jj'}, L_3^{jj'}$ are positive real numbers. Each random field $U_{jj'}$ is then mean-square continuous on $\mathbb{R}^3$ and it power spectral measure has a compact support. Such a model has 63 real parameters $L_1^{jj'}, L_2^{jj'}, L_3^{jj'}$ for $1 \leq j \leq j' \leq 6$ which represent the spatial correlation lengths of the stochastic germs $U_{jj'}$.

(i.2)- Defining an adapted family of functions. The construction of the random field $[\mathbf{G}_0]$ requires the introduction of an adapted family of functions $\{u \mapsto h(\alpha, u)\}_{\alpha>0}$ in which α is a positive real number. Function $u \mapsto h(\alpha, u)$, from $\mathbb{R}$ into $]0\,, +\infty[$, is introduced such that $\Gamma_\alpha = h(\alpha, U)$ is a gamma random variable with parameter α while U is a normalized

Gaussian random variable ($E\{U\} = 0$ and $E\{U^2\} = 1$). Consequently, for all u in $\mathbb{R}$, one has

$$h(\alpha, u) = F_{\Gamma_\alpha}^{-1}(F_U(u)), \tag{8.31}$$

in which $u \mapsto F_U(u) = \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi}} e^{-v}\, dv$ is the cumulative distribution function of the normalized Gaussian random variable U. The function $p \mapsto F_{\Gamma_\alpha}^{-1}(p)$ from $]0\,,1[$ into $]0\,,+\infty[$ is the reciprocal function of the cumulative distribution function $\gamma \mapsto F_{\Gamma_\alpha}(\gamma) = \int_0^{\gamma} \frac{1}{\Gamma(\alpha)}\, t^{\alpha-1}\, e^{-t}\, dt$ of the gamma random variable Γ_α with parameter α in which $\Gamma(\alpha)$ is the gamma function defined by $\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1}\, e^{-t}\, dt$.

(i.3)- Defining the random field $[\mathbf{G}_0]$. For all $\mathbf{x}$ fixed in Ω, the available information defined by Eqs. (8.23) to (8.28), lead us to choose the random matrix $[\mathbf{G}_0(\mathbf{x})]$ in ensemble SG_0^+ defined in Section 2.5.2. Taking into account the properties defined in Section 2.5-(1), (2) and (3), the correlation spatial structure of random field $\mathbf{x} \mapsto [\mathbf{G}_0(\mathbf{x})]$ is then introduced in replacing the Gaussian random variables $U_{jj'}$ by the Gaussian real-valued random fields $\{U_{jj'}(\mathbf{x}), \mathbf{x} \in \mathbb{R}^3\}$ defined in Section 8.2.2-(i.1), for which the correlation spatial structure is defined by a spatial correlation lengths $L_k^{jj'}$. Consequently, the random field $\mathbf{x} \mapsto [\mathbf{G}_0(\mathbf{x})]$, defined on probability space $(\Theta, \mathcal{T}, \mathcal{P})$, indexed by $\mathbb{R}^3$, with values in $\mathbb{M}_6^+(\mathbb{R})$ is constructed as follows:

(1) Let $\{U_{jj'}(\mathbf{x}), \mathbf{x} \in \mathbb{R}^3\}_{1 \le j \le j' \le 6}$ be the 21 independent random fields introduced in Section 8.2.2-(i.1). Consequently, for all $\mathbf{x}$ in $\mathbb{R}^3$,

$$E\{U_{jj'}(\mathbf{x})\} = 0, \quad E\{U_{jj'}(\mathbf{x})^2\} = 1, \quad 1 \le j \le j' \le 6. \tag{8.32}$$

(2) Let δ be the real number, independent of $\mathbf{x}$, such that

$$0 < \delta < \sqrt{7/11} \; < \; 1. \tag{8.33}$$

This parameter which is assumed to be known (resulting, for instance, from an experimental identification solving an inverse problem) allows the statistical fluctuations (dispersion) of the random field $[\mathbf{G}_0]$ to be controlled.

(3) For all $\mathbf{x}$ in $\mathbb{R}^3$, the random matrix $[\mathbf{G}_0(\mathbf{x})]$ is written as

$$[\mathbf{G}_0(\mathbf{x})] = [\mathbf{L}(\mathbf{x})]^T\, [\mathbf{L}(\mathbf{x})], \tag{8.34}$$

in which $[\mathbf{L}(\mathbf{x})]$ is the upper (6×6) real triangular random matrix defined (see Section 2.5.2) as follows:

▷ For $1 \le j \le j' \le 6$, the 21 random fields $\mathbf{x} \mapsto [\mathbf{L}(\mathbf{x})]_{jj'}$ are independent.

▷ For $j < j'$, the real-valued random field $\mathbf{x} \mapsto [\mathbf{L}(\mathbf{x})]_{jj'}$, indexed by $\mathbb{R}^3$,

is defined by $[\mathbf{L}(\mathbf{x})]_{jj'} = \sigma\, U_{jj'}(\mathbf{x})$ in which σ is such that $\sigma = \delta/\sqrt{7}$.
$\triangleright$ For $j = j'$, the positive-valued random field $\mathbf{x} \mapsto [\mathbf{L}(\mathbf{x})]_{jj}$, indexed by $\mathbb{R}^3$, is defined by $[\mathbf{L}(\mathbf{x})]_{jj} = \sigma\,\sqrt{2\,h(\alpha_j, U_{jj}(\mathbf{x}))}$ in which $\alpha_j = 7/(2\delta^2) + (1-j)/2$.

(i.4)- A few basic properties of the random field $[\boldsymbol{G}_0]$. The random field $\mathbf{x} \mapsto [\mathbf{G}_0(\mathbf{x})]$, defined in Section 8.2.2-(i.3), is a homogeneous second-order mean-square continuous random field indexed by $\mathbb{R}^3$ with values in $\mathbb{M}_6^+(\mathbb{R})$ and its trajectories are almost surely continuous on $\mathbb{R}^3$. For all $\mathbf{x} \in \mathbb{R}^3$, one has

$$E\{\|\mathbf{G}_0(\mathbf{x})\|_F^2\} < +\infty, \quad E\{[\mathbf{G}_0(\mathbf{x})]\} = [\,I_6]. \tag{8.35}$$

It can be proven that the newly introduced parameter δ corresponds to the following definition

$$\delta = \left\{\frac{1}{6} E\{\|\,[\mathbf{G}_0(\mathbf{x})] - [\,I_6]\,\|_F^2\}\right\}^{1/2}, \tag{8.36}$$

which shows that

$$E\{\|\,\mathbf{G}_0(\mathbf{x})\,\|_F^2\} = 6\,(\delta^2 + 1), \tag{8.37}$$

in which δ is independent of $\mathbf{x}$. For all $\mathbf{x}$ fixed in $\mathbb{R}^3$, the probability density function with respect to the measure $\widetilde{dG} = 2^{15/2}\,\Pi_{1\leq j\leq k\leq 6}\, d[G]_{jk}$ of random matrix $[\mathbf{G}_0(\mathbf{x})]$ is independent of $\mathbf{x}$ and is written (see Section 2.5.2 with $n = 6$) as

$$p_{[\mathbf{G}_0(\mathbf{x})]}([G]) = \mathbb{1}_{\mathbb{M}_6^+(\mathbb{R})}([G]) \times C_{\mathbf{G}_0} \times \big(\det[G]\big)^{7\frac{(1-\delta^2)}{2\delta^2}} \times \exp\left\{-\frac{7}{2\delta^2}\,\mathrm{tr}\,[G]\right\}, \tag{8.38}$$

where the positive constant $C_{\mathbf{G}_0}$ is defined in Section 2.5.2 with $n = 6$. For all $\mathbf{x}$ fixed in $\mathbb{R}^3$, Eq. (8.38) shows that the random variables $\{[\mathbf{G}_0(\mathbf{x})]_{jk},\ 1 \leq j \leq k \leq 6\}$ are mutually dependent. In addition, the system of the marginal probability distributions of the random field $\mathbf{x} \mapsto [\mathbf{G}_0(\mathbf{x})]$ is completely defined and is not Gaussian. There exists a positive constant b_G independent of $\mathbf{x}$, but depending on δ, such that for all $\mathbf{x} \in \mathbb{R}^3$,

$$E\{\|[\mathbf{G}_0(\mathbf{x})]^{-1}\|^2\} \leq b_G < +\infty. \tag{8.39}$$

Since $[\mathbf{G}_0(\mathbf{x})]$ is a random matrix with values in $\mathbb{M}_6^+(\mathbb{R})$, then $[\mathbf{G}_0(\mathbf{x})]^{-1}$ exists (almost surely). However, since almost sure convergence does not imply mean-square convergence, the previous result cannot simply be deduced. Let $\overline{\Omega} = \Omega \cup \partial\Omega$ be the closure of the bounded set Ω. We then have

$$E\big\{(\sup_{\mathbf{x}\in\overline{\Omega}} \|\,[\mathbf{G}_0(\mathbf{x})]^{-1}\|)^2\big\} = c_G^2 \ < \ +\infty. \tag{8.40}$$