encountered where convection dominates the flow response. The problem arises in representing a convective phenomenon with a static mesh.

4.3.3.1 Problem Definition

Mathematically, the problem is a consequence of instability due to convective domination of the first order hyperbolic equation. The problem can be circumvented by applying any of the following methods: (a) method of characteristics (Huyakorn and Pinder 1983), (b) Eulerian-Lagrangian moving coordinates (Zhang et al. 1993), (c) function transformation (Bai et al. 1994d), and (d) random walk method (Bear and Verruijt 1987). The common nature of these methods is to eliminate the convective term and to transform the transport equation into a more stable parabolic type of equation. Solution of the dispersion-convection equation [e.g., Eq. (4.32)], is particularly challenging as Peclet number, indexing the ratio of convective to diffusive fluxes, increases. At high Peclet numbers, one is usually forced to choose between accepting the presence of nonphysical oscillations within the solution or suffering unwanted numerical dispersion. Of key importance is awareness of the changing nature of the governing equation. Where dispersion dominates [e.g., Eq. (4.205)], the equation is parabolic and causes no particular problem in numerical solution. Where convection dominates [e.g., Eq. (4.204)], the behavior is analogous to a first-order hyperbolic partial differential equation that exhibits a frontal character and creates difficulties in its numerical solution.

Applying the dimensionless concentration and changing only the definition for the dimensionless time t_d in Eq. (4.35) as

$$\begin{cases} c_d = \dfrac{c(x,t)}{c(0,t)} \\ t_d^* = \dfrac{Dt}{L^{*2}} \end{cases} \tag{4.207}$$

Eq. (4.36) can be reformulated as

$$\frac{\partial^2 c}{\partial x_d^2} - \gamma^* \frac{\partial c}{\partial x_d} = \frac{\partial c}{\partial t_d^*} \tag{4.208}$$

with the following initial and boundary conditions:

$$\begin{cases} c(x_d, 0) = 0 \\ c(0, t_d^*) = 1 \\ c(L^*, t_d^*) = 0 \quad \text{(for prescribed concentration)} \\ \dfrac{\partial c(L^*, t_d^*)}{\partial x_d} = 0 \quad \text{(for prescribed flux)} \end{cases} \tag{4.209}$$

Any attempts to solve Eq. (4.208) for a large equivalent Peclet number (EPN),

γ^*, by any numerical technique lead to oscillatory results (Gladwell and Wait 1979).

4.3.3.2 Alternative Methodologies

The most successful technique in eliminating numerical oscillations may, however, be attributed to the application of the upwinding method (Christie et al. 1976) modified from the finite difference iteration. The method is also referred to as upstream weighting (Huyakorn and Pinder 1983). However, despite the utility of this method in reducing spurious oscillations, excessive smearing or numerical dispersion is arbitrarily added to the solutions. Noorishad et al. (1992) provided a review of the effect of the upwind method.

Results for higher dimensional elements in two and three dimensions, where low element continuity is maintained, results in no net improvement (Heinrich et al. 1977). Alternatively, higher order elements using cubic or bicubic Hermitian interpolating functions, together with collocation finite element methods, may concurrently minimize oscillation and smearing (Mohsen 1984; Pinder and Shapiro 1979; van Genuchten and Pinder 1977). However, the computational costs incurred in using higher order elements are high and formulation of the problem often turns out to be cumbersome.

By examining Eq. (4.208), it is known that the first term on the left-hand side of the equation is symmetric in nature, which generally does not induce stability problems in numerical schemes. However, the asymmetry of the numerical formulation, which is reported to cause stability problems, is due to the existence of the other two terms. In consequence, alternative methodologies for minimizing numerical dispersion and oscillation are frequently divided into two groups according to the negative impact due to the existence of: (a) the first term on the right-hand side of Eq. (4.208), and (b) the second term on the left-hand side of Eq. (4.208), respectively. The former is related to the temporal treatment, while the latter is associated with the spatial treatment. The latter method is more popular because it is more effective in reducing the numerical dispersion and oscillation. Occasionally, methods are developed for reducing the negative impacts of both terms in Eq. (4.208) by combined temporal and spatial treatments.

Temporal Treatment

Leismann and Frind (1989) attempted to achieve matrix symmetry by placing the convective term in the previous time level in time marching. The resulting numerical errors are minimized by introducing an artificial dispersion term and by optimal time weighting of all terms on the basis of a Taylor expansion of the governing equation.

Sudicky (1989) indicated that improper selection of a time step size could lead to artificial smearing (i.e., numerical dispersion) or oscillations in the solution. He used Laplace transformation to eliminate the temporal derivative

term from the dispersion-convection equation and solved the ordinary differential equation in the Laplace domain using the conventional Galerkin finite element technique.

Spatial Treatment

In general, the spatial treatment is divided into two groups: moving coordinate methods and function transformation methods.

Moving Coordinate Methods: This method converts a dispersion-convection equation into a hyperbolic-type equation in order to achieve numerical stability (Zhang et al. 1993). Assuming that the medium is fully saturated and that retardation is negligible, the governing equation proposed by Zhang et al. (1993) collapses to the same format as in Eq. (4.208). The classical dispersion-convection transport equation is evaluated in the fixed Eulerian coordinate system.

Using the concept of the total derivative (or Lagrangian derivative), one has

$$\frac{dc}{dt_d^*} = \frac{\partial c}{\partial t_d^*} + \gamma^* \frac{\partial c}{\partial x_d}. \tag{4.210}$$

Substituting Eq. (4.210) into Eq. (4.208), a parabolic-like equation is formed:

$$D\frac{\partial^2 c}{\partial x_d} = \frac{dc}{dt_d^*}. \tag{4.211}$$

The concentration c in Eq. (4.211) no longer represents the concentration at a point in space and in time, but rather the concentration of a fluid particle moving along the characteristic path described by the following equation:

$$\gamma^* = \frac{dx}{dt}. \tag{4.212}$$

This approach is similar in concept and procedure to the "random walk method" in Section 4.2.4 describing stochastic processes, however, with a different explanation. In Eq. (4.211), a Lagrangian moving coordinate system, defined in Eq. (4.210), is used to replace the traditional fixed Eulerian coordinate system. As noted by Zhang et al. (1993), the Lagrangian formulation eliminates the convective term so the governing equation takes on a parabolic format that can be solved more efficiently with a finite element method. The final results are converted back to the Eulerian coordinate system through Eq. (4.210).

In other numerical schemes related to the moving coordinate method, "adaptive grid method" (Hu and Schiesser 1981), "adaptive characteristics method" (Ouyang and Elsworth 1989) and "moving grid method" (Gottardi and Venutelli 1994) are all similar methods to refine the mesh at the steep

concentration front so as to minimize dispersion and oscillation, in comparison with traditional schemes.

Function Transformation Method: Functional transformation is an effective way to convert a parabolic-hyperbolic type equation into a more stable parabolic-type equation. However, the trade-off of this transformation is the result of solving a more challenging time-dependent boundary problem.

- *Method of Ogata and Banks*

  Ogata and Banks (1961) derived an analytical solution of the classic dispersion and convection model of one-dimensional transport using function transformation. This technique has been adopted in solving convection-dominated transport equations.

  Assuming that the concentration is a function of an exponential function, as (Ogata and Banks 1961)

$$c = \Lambda_o(x,t)\exp\left(\frac{vx}{2D} - \frac{v^2 t}{4D}\right) \tag{4.213}$$

  where $\Lambda_o(x,t)$ is a transformation function, then substituting Eq. (4.213) into Eq. (4.32), gives a parabolic-type equation as

$$D\frac{\partial^2 \Lambda_o}{\partial x^2} = \frac{\partial \Lambda_o}{\partial t} \tag{4.214}$$

  with the following initial and boundary conditions:

$$\begin{cases} \Lambda_o = 0 & t = 0 \\ \Lambda_o = c_0 \exp\left(\frac{v^2 t}{4D}\right) & x = 0 \\ \Lambda_o = 0 & x = \infty. \end{cases} \tag{4.215}$$

  It may be noted that the concentration at the inlet boundary becomes time-dependent. Applying Duhamel's theorem (Carslaw and Jaeger 1959) and Laplace transforms to Eqs. (4.214) and (4.215), the following well-known analytical solution is obtained after a variety of clever but tedious analytical maneuvers:

$$c = \frac{c_0}{2}\left[\text{erfc}\left(\frac{x - vt}{2\sqrt{Dt}}\right) + \exp(\frac{vx}{D})\text{erfc}\left(\frac{x + vt}{2\sqrt{Dt}}\right)\right] \tag{4.216}$$

  where "erfc" is the complementary error function, which is related to the error function "erf" through $\text{erfc}(x) = 1 - \text{erf}(x)$.

- *Method of Guymon*

  In solving the convection-dominated transport equation, the presence of numerical oscillations was first recognized by Price et al. (1968). To circumvent this difficulty, Guymon (1970) applied the following functional transformation:

$$c = \Lambda_g(x,t) \exp\left(-\frac{vx}{2D}\right) \tag{4.217}$$

  where $\Lambda_g(x,t)$ is also a transformation function.

  Substituting Eq. (4.217) into (4.32), yields

$$D\frac{\partial^2 \Lambda_g}{\partial x^2} - \frac{v^2 \Lambda_g}{4D} = \frac{\partial \Lambda_g}{\partial t}. \tag{4.218}$$

  In Eq. (4.218), the first-order derivative term with respect to x, which is problematic in representing convection-dominated transport, is eliminated.

  The initial and boundary conditions are transformed to

$$\begin{cases} \Lambda_g = 0 & t = 0 \\ \Lambda_g = c_0 & x = 0 \\ \Lambda_g = c_l \exp\left(\frac{v^2 t}{4D}\right) & x = L^* \end{cases} \tag{4.219}$$

  where c_l is the concentration value at the outlet.

  For zero concentration at the outlet, the outlet boundary condition is simplified to $\Lambda_g = 0$. As a result, the boundary conditions become time-independent. Under such conditions, Guymon's method is simpler than Ogata and Banks' method. However, Guymon's method is not effective in reducing numerical instability at higher Peclet numbers (Guymon 1970).

- *Method of Bai et al.*

  The problems associated with the convection-dominated transport equation are, in general, attributed to the dominance of convection over dispersion. In general, this problem exists for both steady state and transient behavior. Bai et al. (1994d) proposed the following procedure.

  For steady state transport, Eq. (4.208) reduces to

$$\frac{\partial^2 c}{\partial x_d^2} - \gamma^* \frac{\partial c}{\partial x_d} = 0. \tag{4.220}$$

  Assuming

$$\gamma^* = \frac{\partial \Lambda_b}{\Lambda_b \partial x_d} \tag{4.221}$$

where $\Lambda_b(x)$ is an arbitrary function. Eq. (4.221) enables Λ_b to be defined from

$$\Lambda_b = \Lambda_b^0 \exp(\gamma^* x_d) \tag{4.222}$$

where Λ_b^0 is the function Λ_b at its initial value. Similarly, an analog to Eq. (4.220) may be defined as

$$\frac{\partial}{\Lambda_b \partial x_d}\left(\Lambda_b \frac{\partial \Lambda_b^*}{\partial x_d}\right) = \frac{1}{\Lambda_b}\frac{\partial \Lambda_b}{\partial x_d}\frac{\partial \Lambda_b^*}{\partial x_d} + \frac{\partial^2 \Lambda_b^*}{\partial x_d^2} \tag{4.223}$$

where Λ_b^* is another arbitrary function.

From the development of Eqs. (4.221), (4.222), and (4.223), it is amenable to re-write Eq. (4.220) as

$$\frac{\partial}{\partial x_d}\left\{\exp(-\gamma^* x_d)\frac{\partial c}{\partial x_d}\right\} = 0. \tag{4.224}$$

Eq. (4.224) yields the exact form of Eq. (4.220). Since the boundary and initial conditions are unchanged from Eq. (4.209), analytical solution of (4.224) may be easily obtained for the prescribed outlet concentration boundary condition as

$$c = \frac{1}{1 - \exp(-\gamma^*)} + \frac{\exp(\gamma^* x_d)}{1 - \exp(\gamma^*)}. \tag{4.225}$$

For modeling steady state solute transport, the method developed by Bai et al. (1994d) provided more accurate solution in comparison with the analytical solution (van Genuchten 1982) than the Galerkin finite element method (Fletcher 1984) and the upwind weighting method (Huyakorn and Pinder 1983), even at high Peclet numbers.

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Chapter 5

ANALYTICAL SOLUTION

5.1 INTRODUCTION

A mathematical model is a replica of some real-world object or system. It is an attempt to take our understanding of the conceptual process and translate it into mathematical terms. There are many basic ways to solve mathematical equations. Analytical methods represent classical approaches to solve the particular system equations, sometimes in closed form, and sometimes requiring that the final simplified equations are solved numerically. Even though typically constrained to reduced spatial dimensions, simple boundary geometries and initial conditions, analytical methods serve as effective means for preliminary simulation, sensitivity analysis, and benchmark study for numerical validations, due primarily to their convenience and ease of application. In reality, however, closed form analytical solutions are difficult to obtain, and therefore are rare in their application to coupled problems. However, semi-analytical solutions in which numerical inversion is used are useful and popular in the solution of partial differential equations.

Many solutions have already been presented using analytical means, such as function transformations. However, these solutions were provided only as illustrations of some basic or well-known approaches and corresponding results. In contrast, this chapter introduces some popular analytical solution techniques relevant to solving coupled processes. Specifically, three different function transformation techniques, together with a method of differential operators, are presented.

5.2 LAPLACE TRANSFORM

As previously demonstrated, Laplace transform is a powerful and broadly used method for the solution of partial differential equations associated with tran-

sient fluid flow and transport phenomena. The technique removes the time derivative through transformation, rendering partial differential equations as ordinary differential equations. The predominant application of Laplace transforms is in the solution of decoupled or partially decoupled systems.

5.2.1 Flow

For simulation using the dual-porosity concept, Warren and Root's (1963) model may be solved by Laplace transforms only because the fluid pressures in the matrix and the fractures are decoupled in Laplace space. The following evaluations are limited to fluid flow in nondeformable fractured porous media.

5.2.1.1 Solution Method

As an alternative, Barenblatt et al. (1960, 1990) proposed a more complete dual-porosity formulation than those expressed in Chapter 3 by Eqs. (3.83) and (3.84) through considering the cross-phase storage interaction

$$\frac{k_1}{\mu}\nabla^2 p_1 = c_1^* n_1 \frac{\partial p_1}{\partial t} - c_{12}^* n_1 \frac{\partial p_2}{\partial t} + \Gamma(p_1 - p_2) \tag{5.1}$$

$$\frac{k_2}{\mu}\nabla^2 p_2 = c_2^* n_2 \frac{\partial p_2}{\partial t} - c_{21}^* n_2 \frac{\partial p_1}{\partial t} - \Gamma(p_1 - p_2) \tag{5.2}$$

where c_{12}^* and c_{21}^* are the cross-coefficients. The second terms on the right-hand side of Eqs. (5.1) and (5.2) were claimed to have an insignificant impact on fluid mass exchange, and therefore were omitted in the final formulation (Barenblatt et al. 1990), as shown in Eqs. (3.83) and (3.84) (Chapter 3). However, it is understood that the formulation of Eqs. (5.1) and (5.2) is based on the phenomenology of the system. A more rigorously derived counterpart to this equation, represented by Eqs. (3.124) and (3.125), can be rewritten in a similar form to Eqs. (5.1) and (5.2) as

$$\frac{k_1}{\mu}\nabla^2 p_1 = g_1 \frac{\partial p_1}{\partial t} - g_2 \frac{\partial p_2}{\partial t} + \Gamma(p_1 - p_2) \tag{5.3}$$

$$\frac{k_2}{\mu}\nabla^2 p_2 = g_3 \frac{\partial p_2}{\partial t} - g_4 \frac{\partial p_1}{\partial t} - \Gamma(p_1 - p_2) \tag{5.4}$$

where g_1, g_2, g_3, and g_4 correspond to c_{11}^*, c_{12}^*, c_{21}^* and c_{22}^*, respectively, defined in Eq. (3.137) (Chapter 3).

For convenience of comparison, the Warren and Root model in Chapter 3 is rewritten using terms represented in the following two equations, assuming the matrix-fracture flow maintains a quasi-steady state, as

$$0 = g_1 \frac{\partial p_1}{\partial t} + \Gamma(p_1 - p_2) \tag{5.5}$$

$$\frac{k_2}{\mu}\nabla^2 p_2 = g_3\frac{\partial p_2}{\partial t} - \Gamma(p_1 - p_2) \tag{5.6}$$

where the variables and constants have been defined previously.

In contrast, for the model represented by Eqs. (5.3) and (5.4) using the concept of quasi-steady matrix flow Eq. (5.3) should reduce to

$$0 = g_1\frac{\partial p_1}{\partial t} - g_2\frac{\partial p_2}{\partial t} + \Gamma(p_1 - p_2). \tag{5.7}$$

Assuming the following dimensionless quantities for the quasi-steady matrix flow:

$$\begin{cases} P_{Di} = \dfrac{2\pi k_2 h_r (p_0 - p_i)}{q\mu} \quad (i = 1, 2) \\ r_D = \dfrac{r}{r_w} \qquad \lambda_w = \dfrac{\Gamma\mu r_w^2}{k_2} \\ t_D = \dfrac{k_2 t}{\mu\sum\limits_{j=1}^{4} g_j r_w^2} \qquad \omega_j = \dfrac{g_j}{\sum\limits_{j=1}^{4} g_j} \quad (j = 1 \text{ to } 4) \end{cases} \tag{5.8}$$

where h_r is the reservoir thickness, p_0 is the initial reservoir pressure, q is the flow rate at the well, and r_w is the wellbore radius. For a block-type matrix, $\Gamma = [60k_1]/[\mu(s^*)^2]$ where s^* is the average fracture spacing.

More specifically,

$$\begin{cases} \sum\limits_{j=1}^{4} g_j = \dfrac{n_1 + n_2}{K_f} \\ \omega_1 = \dfrac{1}{n_1 + n_2}[(\alpha_3 - n_1 - n_2)\dfrac{K_f}{K_s} + n_1] \\ \omega_2 = -\dfrac{\alpha_3 - n_1 - n_2}{n_1 + n_2}\dfrac{K_f}{K_{fr}} \\ \omega_3 = \dfrac{1}{n_1 + n_2}[(\alpha_3 - n_1 - n_2)\dfrac{K_f}{K_{fr}} + n_2] \\ \omega_4 = -\dfrac{\alpha_3 - n_1 - n_2}{n_1 + n_2}\dfrac{K_f}{K_s} \end{cases} \tag{5.9}$$

where K_{fr}, K_s and K_f are the bulk moduli of fractures, solid grain, and fluid, respectively. Substituting the dimensionless terms given in Eq. (5.8) into the governing equations (5.7) and (5.4), the new formulation for the radial flow is

$$\lambda_w(P_{D2} - P_{D1}) = \omega_1\frac{\partial P_{D1}}{\partial t_D} - \omega_2\frac{\partial P_{D2}}{\partial t_D} \tag{5.10}$$

$$\frac{1}{r_D}\frac{\partial}{\partial r_D}(r_D\frac{\partial P_{D2}}{\partial r_D}) + \lambda_w(P_{D1} - P_{D2}) = \omega_3\frac{\partial P_{D2}}{\partial t_D} - \omega_4\frac{\partial P_{D1}}{\partial t_D}. \tag{5.11}$$