Theoretical Approach to Sand Liquefaction

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Abstract

The paper presents a non associated elasto/viscoplastic constitutive equation for saturated sand. This constitutive equation assumes: instantaneous elastic response, yield surfaces in the sense of viscoplasticity (i.e., after each reloading following with delay in time the loading path), it describes both compressibility and dilatancy and the constitutive equation is non associated (i.e., the viscoplastic potential does not coincide with the yield function). It is shown that in this case a strip in the constitutive domain exists where short-term repeated loading/unloading pulse can produce loosing of stability, i.e., liquefaction.

Introduction.

Very many papers have been devoted to sand liquefaction. Most of these paper are case studies or laboratory tests, but there are also theoretical approaches in which various models have been used (see, for instance, Han and Vardoulakis [1991], Vaid and Sasitharan [1992], Anandarajah [1994], Nemat-Nasser and Shokooh [1980], Veyera and Charlie [1990], besides many others). An excellent review paper presenting laboratory tests and case studies is due to Ishihara [1993]. All main features of sand liquefaction are presented with great accuracy.

In this paper we are presenting a theoretical approach to sand liquefaction. One starts by assuming that, mainly in dynamic problems, saturated sand satisfies an elasto/viscoplastic nonassociated constitutive equation. Thus, the instantaneous response is assumed elastic, in the sense that the two extended body elastic waves (longitudinal and transverse) can propagate in sand. The yield stress can be assumed to be essentially zero. The yield surfaces are surfaces of equal stored (or partially released) energy. The yield surfaces in viscoplasticity are not following

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instantaneously the stress path, but with time delay. Thus, an "instantaneous" loading followed immediately by an 'instantaneous" unloading will not change at all the yield surface. Therefore, a dynamic loading/unloading cycle performed in a very short time interval, will influence very little the yield surface. The constitutive equation formulated for saturated sand is describing both compressibility and dilatancy. Also, it has been found (see literature mentioned by Cristescu and Hunsche [1998]) that only nonassociated constitutive equations can describe accurately the mechanical behavior of saturated sand: thus the yield function does not coincide with the viscoplastic potential. These are the arguments used in the theoretical approach of sand liquefaction:

- Instantaneous elastic response to "instantaneous" loading or unloading;
- the constitutive equation describes both compressibility and dilatancy:
- the constitutive equation is viscoplasticity, i.e., all irreversible deformation is viscoplastic:
- the constitutive equation is nonassociated.

The present analysis was published as a short paragraph in a paper devoted to constitutive equation for sand (Cristescu [1991]) (and sent to the editor of the journal in 1989 already). Since this journal (Int. J. Plasticity) was not dedicated to geomechanics problems, this analysis remained unnoticed by the specialists in the field.

Constitutive equation

The constitutive equation used is of the same kind as used for rocks (Cristescu [1989, 1999], Cristescu and Hunsche [1998]) or for particulate materials (Cristescu [1991, 1996], Cazacu, Jin and Cristescu [1997], Cristescu, Cazacu and Jin [1997], Jin and Cristescu [1998]):

$$D=\frac{\dot{T}}{2G}+\left(\frac{1}{3K}-\frac{1}{2G}\right)\dot{\sigma}\,\mathbf{1}+k\left\langle 1-\frac{W(t)}{H(\mathbf{T})}\right\rangle\frac{\partial F(\mathbf{T})}{\partial \mathbf{T}} \tag{1}$$

where $\mathbf{D}$ is the rate of deformation tensor, $\mathbf{T}$ - the Cauchy stress tensor, σ - the mean stress,

$$W(T)=\int_0^T \sigma(t)\,\dot{\varepsilon}_v^I(t)\,dt+\int_0^T \mathbf{T}'(t):\mathbf{D}'^I(t)\,dt \tag{2}$$

is the irreversible stress work per unit volume, "prime " stands for deviator and ε_v^I is the irreversible volumetric strain. Further in (1) $H(\mathbf{T})$ is the yield function depending on stress invariants and

$$H(T(t)) = W(t) \tag{3}$$

is the equation of stabilization boundary (stabilization of transient creep and/or stress relaxation). Finally $F(\mathbf{T})$ is the viscoplastic potential depending also on stress invariants, and k a viscosity parameter. The irreversible volumetric strain rate is obtained from

$$\dot{\varepsilon}_v = k \left\langle 1 - \frac{W(t)}{H(\mathbf{T})} \right\rangle \frac{\partial F}{\partial \sigma} \tag{4}$$

where the derivative of F is with respect to the mean stress σ. Thus, dilatancy takes place there where $\partial F/\partial \sigma < 0$, while compressibility there where $\partial F/\partial \sigma > 0$. The compressibility/dilatancy boundary is defined by $\partial F/\partial \sigma = 0$.

The first two right-hand side terms in (1) describe the "instantaneous" response, while the last term the "time-effects", or non-instantaneous viscoplastic behavior.

All the above mentioned functions or parameters are determined from a few tests performed in triaxial apparatus. The procedure follows these steps:

First are determined in triaxial tests the **elastic parameters** from short loading/unloading cycles performed at various stress levels, after keeping the stress constant for short period of time (15 – 30 minutes) (see Cristescu [1989], Nawrocki *et al.* [1999]). The reason for this procedure is that if the unloading is performed just after a loading, a significant hysteresis loop is generally observed. If, after performing the loading up to a desired stress level one keeps this stress constant for a short time interval, this hysteresis loop is disappearing during a small unloading/reloading cycle, *i.e.* the rheological properties are "separated" quite well from unloading. Also, by this procedure the "quasistatic" determined elastic parameters are quite close to the "dynamic" ones.

Afterwards is found from tests the **yield function** $H(T)$ without any a priori assumption, by estimating the deformation energy (i.e., $W(t)$). The first right hand term in (2) is the volumetric stress work either stored (in the compressibility domain) or released (in the dilatancy domain). First, the volumetric stress work per unit volume is determined in hydrostatic tests (when the first right-hand side term in (2) is the only one nonzero. Afterwards, both right-hand side terms from (2) are determined in the deviatoric stage of the triaxial tests. Thus is found the yield function involved in (3).

The last step is to find the **viscoplastic potential** by determining the derivatives of F with respect to the stress invariants. That can be done after determining the yield functions $H(\mathbf{T})$ and knowing from tests the orientation of the rate of deformation components along the triaxial loading paths. Why the viscoplastic potential is distinct from the yield function? The viscoplastic potential is defining the orientation of the irreversible strain rate tensor. Therefore according to (4), this function is making precise for what stress state the geomaterial is in a compressible state, for what stress state at is in a dilatant state and finally for what stress state the geomaterial passes from compressibility to dilatancy. The compressibility/dilatancy boundary is therefore defined by $\partial F/\partial \sigma = 0$ (see (4)). If the constitutive law would be associated the equation of this boundary would be just $\partial H/\partial \sigma = 0$. After determining the boundary $\partial H/\partial \sigma = 0$ on can compare it with the compressibility/dilatancy boundary determined from experiment. For sand, as for most geomaterials, the boundary $\partial H/\partial \sigma = 0$ is quite far from the compressibility/dilatancy boundary (see Fig.1). That is why if one wishes to determine accurately the irreversible behaviour of the volume one has to determine in some other way the orientation of the irreversible strain rate tensor, as for instance, by a viscoplastic potential or in a simpler way by a strain-rate orientation tensor (Cristescu and Hunsche [1998]).

In order to determine F(**T**), one starts from the determination of the equations of the C/D boundary from tests; for saturated sand it can be approximated by $-\bar{\sigma}+2f\sigma=0$ with $f = 0.562$. Then $(2f+\alpha)\sigma-\left(1-\frac{\alpha}{3}\right)\bar{\sigma}=0$ is the equation of the short-term failure surface, with $\alpha = 1.34$. The determination of F starts from the determination of the derivative $\partial F/\partial \sigma$. For sand one can use the formula

$$\frac{\partial F}{\partial \sigma} = h_1 \frac{\left(-\bar{\sigma}+2f\sigma\right)\sqrt{\sigma}}{(2f+\alpha)\sigma-\left(1+\frac{\alpha}{3}\right)\bar{\sigma}} \tag{5}$$

where the term $\sqrt{\sigma}$ comes from the matching of the data on sand compressibility in hydrostatic tests. Thus (5) defines quite accurately the passage from compressibility to dilatancy and vice versa, since the equation of the C/D boundary as determined from tests is incorporated in the function F.

For further details of the procedure used to determine the function F see Cristescu [1991] or Cristescu and Hunsche [1998].

For saturated fine silica sand the data by Lade *et al.* [1987] have been used. The yield surfaces H = const. and viscoplastic potential surfaces F = const., as determined from tests, using the procedure described shortly above, and without any a priori assumption concerning their shapes, are shown in Fig.1 (Cristescu [1991]). It is obvious that the two families of surfaces: the yield surfaces H = const. (dotted lines) and viscoplastic potential surfaces F = const. (interrupted lines) are quite distinct. The full line is the short-term failure surface. In Fig.1 $\bar{\sigma}$ is the equivalent stress $\bar{\sigma}=\sqrt{3 II_{T'}}$.

An important feature of the constitutive equation (1) and which will be involved in the theoretical approach to liquefaction is the following. Let us assume

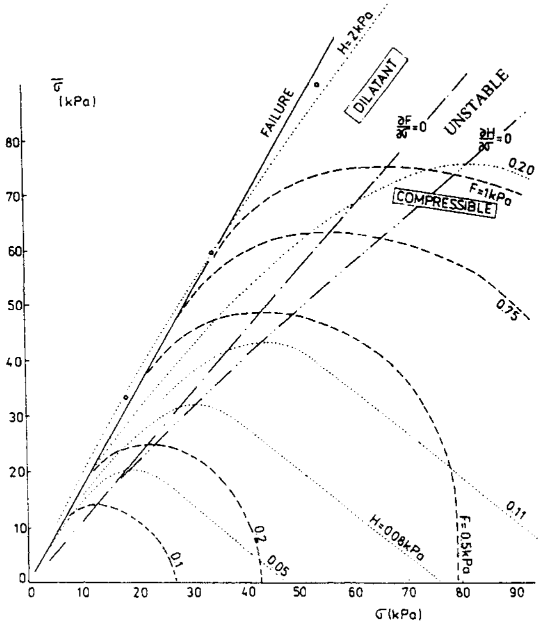

Fig.1 Yield surface H = const, viscoplastic potential surfaces F = const, failure line (full line), compressibility/dilatancy boundary $\partial F / \partial \sigma = 0$ and loosing of stability boundary $\partial H / \partial \sigma = 0$.

that the actual stress states is $\mathbf{T}^P$ and the corresponding value of W in an equilibrium state (i.e., satisfying (3)) is W^P. If at time t_0 the stress state is suddenly changed to $\mathbf{T}(t_0)$ in a loading process (i.e. $H(\mathbf{T}(t_0)) > H(\mathbf{T}^P))$ and is afterwards kept constant, W(t) is varying according to (Cristescu [1989] § 8.4)

$$\frac{W(t)}{H(\mathbf{T}(t_o))} = 1 + \left(\frac{W^P}{H(\mathbf{T}(t_o))} - 1 \right) exp\left[-\frac{k}{H} \frac{\partial F}{\partial \mathbf{T}} : \mathbf{T}(t - t_o) \right]. \qquad (6)$$

Thus the yield surface is following the sudden loading with a significant delay. If, however, the fast loading is followed immediately by a fast unloading (vibratory loading), the change in the yield surface according to (6), is extremely small. This depends on the "relaxation time" which, in our case, is depending on stress state (the reciprocal of $\frac{k}{H}\frac{\partial F}{\partial \mathbf{T}}:\mathbf{T}$ computed for $\mathbf{T}(t_o)$).

A possible theoretical approach to sand liquefaction.

First we would like to mention again that the compressibility/ dilatancy boundary $\partial F/\partial \sigma = 0$ is quite precisely determined, since this boundary, as determined from tests, is incorporated exactly as it is, in the expression of the function F (see (5)). The line (generally not a straight line) $\partial F/\partial \sigma = 0$ is passing by the maxima of the lines $F = \text{const.}$ (see Fig.1). At the right of this curve $\partial F/\partial \sigma > 0$ and the sand is irreversible compressible, i.e., $\dot{\varepsilon}_v^I > 0$, (see (4)). At the left of this curve $\partial F/\partial \sigma < 0$ and the sand is irreversible dilatant $\dot{\varepsilon}_v^I < 0$.

The line $\partial H/\partial \sigma = 0$ is not so precisely determined, since the function $H(\mathbf{T})$ is determined following a long procedure to estimate the deformation energy along the triaxial loading paths. After determining the surfaces $H = \text{const}$ the line $\partial H/\partial \sigma = 0$ is the one uniting the maxima of the curves $H = \text{const.}$ shown in Fig.1. If the constitutive equation would be associated then the line $\partial H/\partial \sigma = 0$

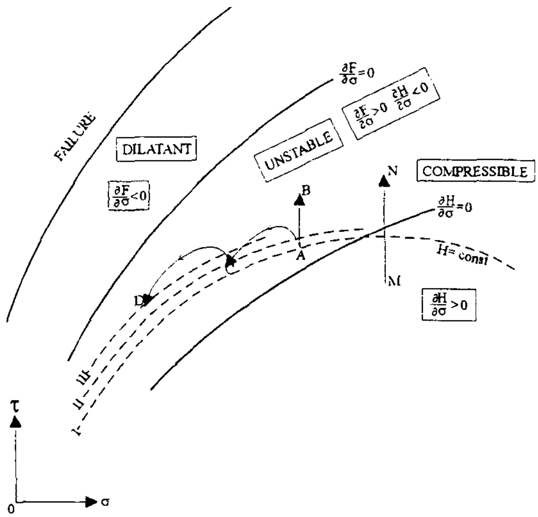

Fig.2 The mechanics of sand liquefaction due to a successive stress vibration.

would be just the compressibility/dilatancy boundary. But the constitutive equation equation is not associated. This line $\partial H/\partial\sigma = 0$ has another meaning which will be revealed below. It will be shown that in the domain where both inequalities

$$\frac{\partial F}{\partial \sigma} > 0 \quad , \frac{\partial H}{\partial \sigma} < 0 \tag{7}$$

are simultaneous satisfied, a saturated sand may become unstable if subjected to several dynamic loading/unloading pulses (Cristescu [1991]). That is shown schematically in Fig.2 . This figure shows a portion of the strip bounded between the two lines $\partial F/\partial\sigma = 0$ and $\partial H/\partial\sigma = 0$ shown in Fig.1. This is a strip of the "compressibility" domain $(\partial F/\partial\sigma > 0)$ where however $\partial H/\partial\sigma < 0$, i.e., the slopes of the H = const. curves shown in Fig.1 have a normal which, if projected on the σ axis is oriented towards the negative direction.

Let us assume that the stress state existing at a certain location in the sand mass (under a building, say, etc.) is represented by point A belonging to the strip (7) (Fig.2) and that this is an equilibrium state. If a dynamic loading/unloading stress pulse is superposed the sand may not be able to carry any more this load. For simplicity let us assume that the stress vibration is a shearing loading increasing the octahedral shear stress τ only, for instance along AB. If the sand is saturated such variation (i.e. only of τ) is not possible. When τ begins to increase, since this loading path belongs entirely to the compressibility domain, the solid skeleton is subjected to a dynamic compressibility. That will produce an increase of the pore pressure, which in turn will reduce the mean stress in the skeleton. Thus each such pulse will produce a decrease of the mean stress in the skeleton. On the other hand since the pulse is of short duration the initial yield surface (position I in the Figure 1) will move (change) very little, during this pulse, to the position II, say. Thus the real loading stress trajectory is AC instead of AB. The whole segment AC means loading. The final stress state may be located even under the yield surface shown as II. The second pulse, if it is reaching points outside the yield surface II will produce an additional decrease of the mean stress in the skeleton. Due to the peculiar shape of the yield surfaces H = const. in this strip (7), each such pulse will produce a decrease of the mean stress and, maybe, that of τ needed to produce yielding. After several such pulses the stress under a building, say, will be too small to be able to carry out the loading. If during this loading process the stress state reaches the domain $\partial F/\partial\sigma < 0$, the sand may again become stable, and further loading may significantly increasing both σ and $\bar{\sigma}$.

The stress states in the domain $\partial H/\partial\sigma > 0$ are stable states. However, if the dynamic loading is producing a significant increase of τ (see segment MN in Fig.2), then during this dynamic loading the stress states may still reach states in the strip (7). In this case, repeated pulses during which stress states in the strip (7) or reached, may still produce a loosing of stability, due to the same mechanism.

Experimental evidence

Very nice experimental data obtained by Lade [1993, 1994a,b] have shown that instability of saturated sand in the domain (6) is possible in cyclic loading. The liquefaction was obtained in the laboratory and has taken place in a similar way as described above. This shows that a non-associated viscoplastic model where a compressibility domain exists between the lines $\partial F/\partial \sigma = 0$ and $\partial H/\partial \sigma = 0$ can explain why loosing of stability can take place. In this analysis it is essential to take into account that the viscoplastic yield surfaces are changing very little during a short time loading/unloading pulse. This is impossible in the framework of time-independent plasticity, where the yield surfaces are moving simultaneously with the loading path. For other results concerning saturated sand instability see Zlatovic and Ishihara [1997], Tsukamoto et al. [1998], Uathyakumar [1996], besides others.

Conclusion

A nonassociated elasto/viscoplastic constitutive equation formulated for saturated sand can explain why repeated dynamic loadings can liquefy the sand.

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Fundamental Dynamic Behavior of Foundations on Sand

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Abstract

Despite numerous past attempts and research efforts, comparisons of physical test results on the dynamic interaction of foundations with sandy soils with analytical theories have continued to be a subject of considerable debate and inconsistency. To determine the source of the difficulties that besiege the physical problem, a systematic investigation was performed using the centrifuge scaled modeling method to examine at a fundamental level the dynamic behavior of surface foundations under both vertical and horizontal excitations. Finally exposed by the experimental approach, a basic reason for the continuing difficulty in characterizing the dynamic behavior of foundations on sandy soils is discovered to be a key incompatibility of present analytical frameworks with the observed physical soil-foundation behavior. While a deeper physical understanding is required prior to a proper theoretical resolution, a conceptually new but practically simple analytical framework which can explicitly recognize and rationally accommodate the granular soil dynamics problem is found to be possible using the instrument of *Impedance Modification Factors.*

Introduction

The key to assessing the effects of dynamic soil-structure interaction on buildings, bridges and superstructures is a reliable determination of the dynamic response of the foundation under general multi-directional loading. As basic a problem as it can be in soil dynamics, the vibratory characteristics of foundations on granular soils have remained a frustrating mystery for both researchers and practitioners for many years. Physically, a major source of the difficulties probably lies in the complex stress-strain relationship of such deposits under cyclic loading and spatially inhomogeneous stress conditions (e.g., Hardin and Drnevich 1972). Analytically, the solution of the related three-dimensional

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